A TREATISE

ON

PLANE CO-ORDINATE GEOMETRY;

OR, THE

APPLICATION OF THE METHOD OF CO-ORDINATES

TO

THE SOLUTION

OF

PROBLEMS IN PLANE GEOMETRY.

PART I.

BY THE REV. M.^thew. O'BRIEN, 1814-1855

PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY IN KING'S COLLEGE, LONDON, AND LATE FELLOW OF CAIUS COLLEGE.

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PREFACE.

THE subject treated of in the following pages is usually styled *Analytical Geometry*, but its real nature seems to be better expressed by the title *Co-ordinate Geometry*, since it consists entirely in the application of the Method of Co-ordinates to the Solution of Geometrical Problems.

The present Treatise, in which we shall confine our attention to figures and curves in one plane, will consist of two parts: the first part is all that is at present published; it contains the application of the method of Co-ordinates to Right Lines, Circles, and Conic Sections. In the second part, the properties of Curves in general, with reference to Tangents, Asymptotes, Singular Points, Curvature, &c., will be investigated, without assuming a knowledge of the Differential Calculus on the part of the Student. A Historical Account of the subject, and a large collection of Problems will be added.

The complete analysis given in the Table of Contents renders it unnecessary to say much here respecting the plan pursued in the Treatise. In the first chapter the meaning of the signs = + and −, and the nature of negative and imaginary quantities, are fully explained, on principles which seem to combine generality and simplicity. The difficulties

which are supposed to beset the foundations of Algebra, are partly due to the indistinctness of the definitions usually given of algebraical symbols. In the subsequent chapters, the Author has endeavoured to adhere to a uniformity of method which, he hopes, will render it easy to acquire and retain a knowledge of the subject. He has also made use of symmetrical equations as much as possible, as there can be no doubt that many advantages are lost, and none gained, by want of attention to symmetry in analytical processes. The properties of Conjugate diameters are investigated by means of the angle called the Eccentric Anomaly in Astronomy, but the same properties are deduced without making use of this angle in Chapter XI. Several geometrical illustrations are given in notes, and various Problems and Examples, many of which are taken from the Senate-House Problems and other sources are added at the end of each chapter.

CAMBRIDGE,
March, 1844.

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ERRATA.

Page 5, third line of Art. 13,

instead of $(X - Y) + Y = 0$, and $(Y - X) + X = 0$,
read $(X - Y) + Y = X$, and $(Y - X) + X = Y$.

Page 54, last two lines of Art. 141, *instead of* and in the third inside it, *read* and in the third it may be inside it.

Page 97, Art. 232, properly speaking, instead of saying that

$$PR = PR', \quad \frac{TQ^2}{TA} = 4m', \quad \text{and } TA = h;$$

we should have said that

$$PR = -PR', \quad \frac{TQ^2}{TA} = -4m', \quad \text{and } TA = -h.$$

In fact we have considered PR' and TA without regard to sign.

Page 123. The same remark applies to Art. 304; we should have said that $CP = -CP'$, $MP' = -(a + x)$, $CD = -CD'$, and $MQ = -MQ'$.

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PLANE CO-ORDINATE GEOMETRY.

CHAPTER I.

MEANING OF THE SIGNS $= + -$. NATURE OF NEGATIVE QUANTITIES. MULTIPLICATION BY A NEGATIVE QUANTITY. OF IMPOSSIBLE QUANTITIES, AND OF THE SIGN $-^{\frac{1}{2}}$. SENSE IN WHICH THE SIGNS $= + -$ ARE USED IN THE PRESENT TREATISE.

Meaning of the signs $= + -$.

Before we enter directly upon the subject of the present treatise, it will be necessary to make a few observations respecting some of the principles of Algebra; chiefly with a view tc explain the nature of negative quantities, and the use of the sign - in Co-ordinate Geometry. We shall commence with the following definitions.

1. The notation, $X = Y$, is a short way of expressing the proposition, *X is equivalent to Y;* the sign = being an abbreviation for the words "*is equivalent to.*"

By the word "*equivalence*" is meant "*sameness in certain particulars supposed to be understood.*" Thus, when we say that 42 shillings are equivalent to 2 guineas, we mean, that 42 shillings are the same as 2 guineas in a certain particular supposed to be understood, namely, in *value;* but not in weight or magnitude. Hence the notation $X = Y$ may have a variety of significations according to the sense in which we use the word equivalent. Of course in every investigation the precise meaning of this word is supposed to be settled and understood.

2. The notation, $X+Y$, is a short way of expressing the words, X *together with* Y: the sign + being an abbreviation for the words "*together with.*"

3. The notation, $X-Y$, is used to represent that thing, whatever it be, which together with Y is equivalent to X: or, in symbolical language, $(X-Y)+Y=X$.

The words "*together with*" are very general and may be used in a variety of different senses: therefore the sign + may have various meanings; and the same is true of the sign −, which, by the definition just given, has a signification depending upon that of +. The precise meaning of these signs, as in the case of =, is supposed to be settled and understood in every investigation.

4. The following examples will shew the nature of these signs thus defined, and some of the different senses in which they may be used,

$5+3=8$ .... (1),

$5-3=2$ .... (2),

£5 loss + £3 gain = £2 loss .... (3),

£5 loss − £3 gain = £8 loss .... (4),

£5 loss − £8 loss = £3 gain .... (5),

3 miles travelled eastward + 2 westward = 1 eastward....(6),

3 miles travelled eastward − 2 westward = 5 eastward....(7),

3 miles travelled eastward − 5 eastward = 2 westward....(8),

£1 + 1 cwt. = 20*s*. + 112 lbs.................(9).

Equation (1) translated into ordinary language may be expressed thus: 5 together with 3 is equivalent to 8; therefore + here denotes common addition, and = denotes absolute sameness.

Using + and = in this sense, we have, $2+3=5$; therefore, by the definition of $X-Y$, 2 is the value of $5-3$, and the truth of the equation (2) is manifest. It is clear that − in this case denotes ordinary subtraction.

In equation (3) + denotes that the loss and gain affect the same property, and = denotes sameness so far as the effect produced on that property is concerned; and thus the equation expressed in ordinary language means, that a man's losing £5 and gaining £3 is the same thing as his losing £2 so far as his property is concerned.

Using + and = in this sense, the truth of the equations (4) and (5) is evident from the definition of $X - Y$; for £8 loss + £3 gain = £5 loss, therefore, by the definition, (4) is true; also £3 gain + £8 loss = £5 loss, and therefore, by the definition, (5) is true.

In equation (6) + denotes that the traveller commences the 2 miles westward as soon as he gets to the end of the 3 miles eastward, and = denotes sameness so far as his final distance from his starting point is concerned: and thus in ordinary language the equation means, that a man's travelling 3 miles eastward and then 2 miles westward is the same thing as his travelling 1 mile eastward, so far as his final distance from his starting point is concerned.

Using + and = in this sense, the truth of the equations (7) and (8) follows immediately from the definition of $X - Y$, as in the case of (4) and (5).

In equation (9) + has merely the force of the particle *and*, and = signifies that the things on the first side of the equation are respectively or separately equivalent to those on the second side; and thus in ordinary language the equation means, that £1 and 1 cwt. are respectively equivalent to 20*s*. and 112 lbs. This explains the meaning of such an equation as

$$a + b\sqrt{(-1)} = c + d\sqrt{(-1)}.$$

5. It is evident that 5 + 3, £5 loss + £3 gain, and 3 miles eastward + 2 westward are the same things, respectively, as 3 + 5, £3 gain + £5 loss, and 2 miles westward + 3 eastward; and in general that $X + Y$ is the same thing as $Y + X$. At least, we shall never attach any meaning to the words "*together with*" for which it is not true that X together with Y is the same thing as Y together with X.

6. The 5th and 8th of the above examples serve an important purpose; they shew that the definition we have given of $X - Y$ applies, without any difficulty whatever, when X is a magnitude of the same kind as, and numerically less than, Y; and that we are therefore at liberty to suppose X to be of any magnitude we please compared with Y. This observation is important, as we are now about to use a notation which would be inadmissible, if

there were any restriction implied in the definition of $X - Y$ requiring X to be a magnitude numerically greater than Y.

Of Negative Quantities.

In various parts of mathematics we often meet with two quantities of such a nature, that, when taken together, they are equivalent to zero. Thus £1 loss and £1 gain, taken together, are equivalent to zero (using the words "taken together" and "equivalent" in the same sense as in Ex. (3) Art. 4). Again, using these words as in Ex. (6), 2 miles travelled eastward and 2 westward, taken together, are equivalent to zero. And a variety of other instances might be given, such as, two equal and opposite forces or velocities, vitreous and resinous electricities, &c. Hence, in order to distinguish and represent two quantities thus related, it becomes desirable to adopt a special nomenclature and notation, which we now proceed to explain.

7. When two things taken together are equivalent to zero, each is said to be the *negative* of the other; or, in symbolical language, if $X + Y = 0$, X is said to be the *negative* of Y, and Y the *negative* of X.

8. It is usual to select either of the two things thus related and call it *positive*, and then the other is said to be *negative*: thus we may call gain positive and loss negative; or, if we please, we may call loss positive and gain negative. We use the term positive, therefore, not to express any peculiarity in the nature of a quantity (except it be that it is capable of having a negative), but merely to mark, for convenience sake, one of the two quantities which taken together are equivalent to zero.

9. We arrive at a very excellent notation for representing the negative of a quantity in the following manner.

By the definition of $X - Y$ above given, it appears that $0 - Y$ is that quantity which together with Y is equivalent to zero; hence, by the definition of a negative, $0 - Y$ is the negative of Y. Now, in such expressions as $0 + Y$ and $0 - Y$, it is usual to omit the symbol 0, and to write them simply in the form $+Y - Y$; and thus $-Y$, instead of $0 - Y$, is employed to represent the

negative of Y. In this manner it is that our definition of $X - Y$ leads to a method of denoting the negative of a quantity (see the observation made in Art. 6), and to the use of the sign $-$ as an independent sign.

We may regard $-$ in this point of view as an abbreviation for the words "*the negative of*"; so that $-Y$ expressed in ordinary language means, *the negative of* Y.

10. $0 + Y$ is evidently the same thing as Y, and therefore, if we omit the symbol 0, $+Y$ and Y mean the same thing. There does not however seem to be any advantage gained by making this independent use of the sign $+$, except in a few cases where it avoids circumlocution, such as the common notation $\pm \sqrt{(X)}$.

11. If we bear in mind that $-Y$ is merely a simple way of writing $0 - Y$, which, according to our definition, denotes that quantity which together with Y is equivalent to zero, there will be no difficulty in perceiving the truth of the various propositions respecting the sign $-$ usually given in treatises on Algebra; for instance, the following:

12. To shew that $-(-Y) = Y$.

Since $(0 - Y) + Y = 0$, by our definition of $X - Y$, it follows, from the definition of a negative, that Y is the negative of $(0 - Y)$; and therefore that $Y = 0 - (0 - Y)$; or, omitting the zeros, that $Y = -(-Y)$. Q. E. D.

13. To shew that $-(X - Y) = Y - X$.

By our definition of $X - Y$, we have

$$(X - Y) + Y = 0, \text{ and } (Y - X) + X = 0;$$

therefore, adding these equations, and taking away from each side of the result the common quantity $X + Y$, we find

$$(X - Y) + (Y - X) = 0.$$

Hence, by the definition of a negative, we find that $(Y - X)$ is the negative of $(X - Y)$, and therefore

$$Y - X = 0 - (X - Y) = -(X - Y). \quad \text{Q. E. D.}$$

14. To shew that, to add $-Y$ to any quantity X, is the same as, to subtract Y, and, to subtract $-Y$, is the same as, to add Y.

Since $(0 - Y) + Y = 0$, it follows that

$$X + (0 - Y) + Y = X;$$

and therefore, by the definition of $X - Y$, we have

$$X + (0 - Y) = X - Y,$$

which shews that X together with $-Y$ is the same thing as $X - Y$. Q. E. D.

Again, since $X + Y + (0 - Y) = X$, we have, by the definition of $X - Y$,
$$X + Y = X - (0 - Y);$$
which shews that to subtract $-Y$ is the same thing as to add Y. Q. E. D.

Of Multiplication by a Negative Quantity.

15. To determine the meaning of multiplication by a negative quantity we must proceed very much in the same manner we do in the case of negative or fractional powers. The original definition of a^m necessarily supposes that m is a positive integer; but, in order to extend the meaning of this notation, we assume, that all the consequences which follow from the original definition, and which *may* hold equally well whether m be an integer or not, are true in general. Thus the law, $a^m a^n = a^{m+n}$, follows from the original definition, and is capable of holding when m and n are not integers: we therefore assume it to be true in general, and it determines the meaning of a^m when m is negative or fractional.

16. In exactly the same way, the definition of $a.b$ originally given supposes the factors to be integers; but from this definition it follows that $(a + c).b = a.b + c.b$ and that $a.(b + c) = a.b + a.c$; which laws are capable of holding when a, b, and c are not integers. We therefore assume these laws to be true in general, and they serve to determine the meaning of a product when the factors are negative or fractional, as we now proceed to shew.

17. To determine the meaning of $(-a).b$, $a.(-b)$, and $(-a).(-b)$.

By the assumed laws we have
$$\{(0 - a) + a\}.b = (0 - a)b + a.b;$$
but $(0 - a) + a = 0$, and $0 - a$ is the same thing as $-a$; therefore, since $0.b = 0$,* we have
$$0 = (-a).b + a.b, \text{ and } \therefore (-a).b = -(a.b).$$

* That, $0.b = 0$, follows from the original definition of $a.b$.

Hence it appears that the result of multiplying b by $-a$ is the negative of $a.b$.

In just the same way we may shew that $a.(-b) = -(a.b)$.

Also, by the results just obtained, we have

$(-a)(-b) = -\{a.(-b)\} = -(-a.b) = a.b$, by Art. 12.

Hence the product of $-a$ and $-b$ is $a.b$.

18. To determine the meaning of $\left(\frac{m}{n}\right).\left(\frac{p}{q}\right)$.

From the assumed law, $(a+c).b = a.b + c.b$, it easily follows that, $(a + c + d + e + \&c....)\,b = a.b + c.b + e.b + d.b + \&c.$. therefore we have

$$\left(\frac{m}{n} + \frac{m}{n} + \frac{m}{n} \ldots \text{to } n \text{ terms}\right).\frac{p}{q} = \frac{m}{n}.\frac{p}{q} + \frac{m}{n}.\frac{p}{q} \ldots \text{to } n \text{ terms},$$

$$\text{or } m.\frac{p}{q} = n.\left(\frac{m}{n}.\frac{p}{q}\right) \ldots\ldots\ldots\ldots (1);$$

and in exactly the same way we may shew that

$$m.p = q.\left(m.\frac{p}{q}\right) = q.n.\left(\frac{m}{n}.\frac{p}{q}\right) \text{ by (1)}.$$

Therefore, dividing by $q.n$, we have

$$\frac{m}{n}.\frac{p}{q} = \frac{m.p}{n.q}.$$

Hence the product of two fractions is the faction, whose numerator is the product of the numerators, and denominator the product of the denominators.

Of Impossible Quantities, and the Sign $-^{\frac{1}{2}}$ or $\sqrt{(-1)}$.

19. It appears from Art. 17 that there is no positive nor negative quantity whose square is a negative quantity: the square root of a negative quantity, therefore, is neither positive nor negative. Now suppose that, in any investigation, all the quantities we are engaged with must be either positive or negative, at least, that we agree to consider no other kind of quantity: then, if any of our operations should lead to a result in the form of $\sqrt{(-a)}$, we must reject this result, and regard the operations that lead to it as impossible to be performed. In

this point of view the square root of a negative quantity may be strictly called impossible or imaginary.

20. In the first part of the present treatise we shall consider all the quantities we are engaged with to be either positive or negative, because there will be no necessity to introduce any other kind of quantity; on the contrary, the simplicity of the subject would be quite lost, and no advantage would be gained, if we were to admit any quantities besides simply positive and negative quantities. This being settled, we must reject the square root of a negative quantity whenever it occurs, and regard the operations that lead to it as impossible.

21. But the square root of a negative quantity is impossible *only* in this point of view, for we shall now shew that there is a quantity whose square is a negative quantity.

22. $-Y$ represents the result of a certain operation performed upon Y, namely, the operation of converting a quantity into its negative; $-(-Y)$ denotes the result of performing this operation upon $-Y$, *i.e.* twice successively upon Y; $-\{-(-Y)\}$ the result of performing it three times successively upon Y, and so on. To represent these results of repeating the operation more concisely and generally, we shall denote them respectively by $-^2 Y$, $-^3 Y$, and so on; and, in general, we shall assume $-^m Y$ to denote the result of performing the operation m times successively upon Y.

In like manner, if we suppose $\dashv Y$ to denote the result of any other operation of any kind performed upon Y, we shall assume $\dashv^m Y$ to denote the result of performing it m times successively upon Y.

23. This definition of $\dashv^m Y$ supposes m to be an integer, but we may extend the meaning of the notation, in the same manner as we do in the case of ordinary powers, by supposing that all the properties which follow from the original definition, and are capable of holding when m is not integral, are true in all cases, whatever m may be. For instance, it follows from the original definition that $\dashv^m(\dashv^n Y) = \dashv^{m+n} Y$, and this property is capable of holding when m and n are not integral; we shall therefore assume it to be true in general.

Hence, to determine the meaning $\dashv^{\frac{1}{2}} Y$ we have only to put $m = n = \frac{1}{2}$ in the assumed property, and we find

$$\dashv^{\frac{1}{2}}(\dashv^{\frac{1}{2}} Y) = \dashv Y,$$

which shews that $\dashv^{\frac{1}{2}}$ represents any operation which, performed twice successively on any quantity, gives the same result as the operation $\dashv$ performed once.

24. Hence $-^{\frac{1}{2}}$ represents any operation which, repeated twice successively upon a quantity, converts it into its negative. On this account I shall venture to call $-^{\frac{1}{2}}$ the *seminegative** sign, and $-^{\frac{1}{2}} Y$ the *seminegative* of Y. Hence the seminegative of the seminegative of Y is the negative of Y; or, in other words, the same operation which converts Y into its seminegative, converts the seminegative of Y into the negative of Y.

25. For example, by turning a traveller's direction through a right angle from right towards left, a mile eastward is changed into a mile northward; and, by exactly the same change, a mile northward is changed into a mile westward; but a mile westward is, as we have seen, the negative of a mile eastward; hence it follows that a mile northward is the seminegative of a mile eastward, and if we represent the former by Y the latter is represented by $-^{\frac{1}{2}} Y$.

26. We have now obtained a quantity whose square is a negative quantity, for it is easy to shew that $(-^{\frac{1}{2}} Y).(-^{\frac{1}{2}} Y) = -Y^2$ in the following manner.

It follows, from the original definition of $-^m Y$, and from the nature of multiplication by a negative quantity explained in Art. 17, that $(-^m X).(-^n X) = -^{m+n}(X.Y)$; and this law is capable of holding when m and n are not integral; we may therefore

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* The sign $-^{\frac{1}{2}}$ or $\sqrt{(-1)}$, as it is commonly written, seems to me to require some specific name to indicate the reality of the operation it represents. Besides, since

$$(-1)^{\frac{1}{m}} = \cos\frac{\pi}{m} + \sin\frac{\pi}{m}\sqrt{(-1)},$$

all the roots of $-$ may be expressed very simply by means of its square root; and therefore the square root may be advantageously distinguished from the other roots by a special name.

assume it to be true in general. Now, putting $m = n = \frac{1}{2}$ and $X = Y$, we find immediately,

$$(-^{\frac{1}{2}} Y) . (-^{\frac{1}{2}} Y) = -Y^2. \qquad \text{Q. E. D.}$$

Hence it appears that the square of the seminegative of Y is the negative of Y. (See Art. 21.)

We shall return to the consideration of seminegative quantities in the second part of this treatise.

Meaning of the Signs $= + -$ *in Co-ordinate Geometry.*

27. In the following pages we shall generally employ the signs $= + -$ in the same manner as in the examples of distances performed by a traveller eastward or westward, given in Art. 4, Ex. (6), (7), (8); for we shall, for the most part, be engaged with the consideration of similar distances, which are supposed to be traced by the motion of a point in particular directions.

Let A, a, a', a'', &c. represent any distances described along a right line AB (fig. 1) by a tracing point moving either from right to left or from left to right; and, to avoid circumlocution, let us call the position of the tracing point, when it commences describing any distance, the *beginning* of that distance; and its position when it has completed it, the *end* of the distance. Then, by the equation

$$a + a' + a'' + a''' + \&c. = A,$$

we shall always mean, that the end of the distance a is the beginning of a', the end of a' the beginning a'', the end of a'' the beginning of a''', and so on; and that when the tracing point has described the compound distance $a + a' + a'' + \&c.$ (beginning a' at the end of a, a'' at the end of a', and so on), its ultimate position is the same as if it had simply described the distance A; or, in other words, that the beginning and end of the compound distance $a + a' + a'' + \&c.$ coincide respectively with those of A.

28. Having thus settled the meaning of the signs $=$ and $+$, that of the sign $-$ may be determined in the following manner.

Let a and b represent two distances equal in magnitude, the former being described from left to right, the latter from right to left; then, using $=$ and $+$ in the sense just settled, it is evident

that $b + a = 0$; therefore, by the definition of a negative, b is the negative of a, and is represented by $0 - b$ or $-b$, as we have shewn (see Art. 9). Hence it appears that if a represent any distance described from left to right, $-a$ denotes a distance equal to it in magnitude, but described from right to left; and the reverse is also true.

29. Since $a - b = a + (-b)$ (see Art. 14), it follows that $a - b$ represents the final distance of the tracing point from its original position, when it has described the distance a, and then the distance $(-b)$, which latter denotes a distance measured equal to b in magnitude, but opposite to it in direction. And this is true whatever distances a and b may represent.

30. It appears hence, from Art. 8, where we have explained the meaning of the term positive, that if we regard distances measured from left to right as positive, those measured from right to left will be negative; or, if we choose to call the latter distances positive, the former will be negative.

31. All that we have said respecting distances described from right to left or from left to right applies equally well to distances described along any line in any position, whether it be a right line or a curved line. The same may also be said of angular distances described by a revolving line.

These few observations respecting the signs $=$ $+$ and $-$ suffice for our present purpose, and we shall now proceed to explain the nature and object of Co-ordinate Geometry.

CHAPTER II.

NATURE OF CO-ORDINATE GEOMETRY. REPRESENTATION OF POINTS BY MEANS OF CO-ORDINATES. PROPOSITIONS RESPECTING THE CO-ORDINATES OF POINTS. REPRESENTATION OF CURVES BY MEANS OF EQUATIONS BETWEEN x AND y.

32. By *Co-ordinate Geometry* we mean that method or system invented by Descartes, in which the positions of points are determined, and the forms of curves and surfaces defined and classified, by means of what are called co-ordinates. In the present treatise we shall confine our attention to one plane, and it is on this account that we have given the name of *Plane* Co-ordinate Geometry to the subject. We shall suppose that all the points, lines, &c. we have occasion to consider, lie in the plane of the paper, except in certain cases, where it will be necessary to make use of solid figures.

Method of representing the Position of a Point by means of Co-ordinates.

33. Let OX, OY (fig. 2) be two right lines perpendicular to each other, and P any point: draw PM perpendicular to OX and PN to OY. Then OM and ON are called the *co-ordinates* of the point P with reference to the lines OX, OY, which are termed the *co-ordinate axes*. It is clear that these co-ordinates serve to determine the position of P with respect to OX and OY; for if OM and ON be given, the lines MP and NP are also given, and therefore their point of intersection, which is P, is known in position with respect to OX and OY.

34. The lines OM and ON acquired the name of co-ordinates in the following manner. From any series of points P, P', P'', &c. (fig. 3), suppose that we drop perpendiculars

PM, $P'M'$, $P''M''$, &c. upon a given right line OX; then, if these perpendiculars, and the portions OM, OM', OM'', &c. which they cut off from the given line, be given in order and magnitude, the points P, P', P'', &c. are determined in order and position. On this account the perpendiculars were called *ordinates* (from *ordino*, which signifies to arrange in order or succession), and the portions OM, OM', &c. cut off from OX were called the *abscissæ* of the ordinates. Now, if we draw OY at right angles to OX, and drop the perpendiculars PN, $P'N'$, $P''N''$ upon it, these perpendiculars, which are the ordinates of the points with respect to OY, are respectively equal to the abscissæ, and may be used in place of them: and thus the ordination, so to speak, of a series of points, may be effected by a double system of ordinates instead of a system of ordinates and abscissæ. In this point of view the lines PM and PN, or OM and ON if we please, are naturally called the *co-ordinates* of the point P. Thus it was that the lines OM and ON came to be designated by the term *co-ordinate.* The ordinates of a series of points were called by Newton "*lineæ ordinatim applicatæ*," and by some authors the ordinate of a point was termed "*crus efficiens*," and the abscissa "*crus patiens*:" but the name co-ordinate is now universally used.

35. Any distance described from O along OX is generally denoted by the letter x, and any distance along OY by the letter y. On this account OX is called *the axis of* x, and OY *the axis of* y. The point O, where the two axes meet, is called *the origin.* All distances described to the right along OX, or upwards along OY, we shall consider to be *positive;* and then all distances described to the left along OX, or downwards along OY, will be *negative.* The principles upon which we do this are explained in Arts. 27–31. We shall call OX the positive axis of x, OX' the negative axis of x, OY the positive axis of y, and OY' the negative axis of y. The point whose co-ordinates are x and y we shall term, the point (xy), to avoid circumlocution.

36. Hence if a and b be any two positive distances, *i. e.* distances described by a tracing point moving along OX or OY in

the positive direction, and if $x = a$ and $y = b$, the point (xy) is determined by describing OM and ON equal in magnitude to a and b, as in (fig. 4), and then the intersection (P) of the perpendiculars MP and NP will be the point (xy).

If $x = -a$ and $y = b$, the distance OM must be described in the negative direction, by Art. 28, and therefore (fig. 5) represents the proper position of P.

If $x = a$ and $y = -b$, ON must be described in the negative direction, and therefore (fig. 6) represents the position of P.

If $x = -a$ and $y = -b$, both OM and ON must be described in the negative direction, therefore (fig. 7) represents the position of P.

We may perceive, hence, the important use of negative quantities in the present subject; without them it would be necessary to assume one set of letters to represent distances described in the positive directions, and another set to denote those described in the negative directions.

37. The following propositions will serve to illustrate this method of employing co-ordinates to represent the positions of points: the first of them is of considerable importance, and we shall often have occasion to refer to it.

PROP. I.

38. P and P' being any two points referred to co-ordinate axes OX, OY (fig. 8), to determine what relations subsist between the co-ordinates of P and P', the distance PP', and the angle which PP' makes with OX.

Let xy and $x'y'$ be the co-ordinates of P and P', draw PM, $P'M'$ perpendicular to OX, and PQ parallel to OX: then $PQ = MM' = OM' - OM = x' - x$, $P'Q = P'M - PM = y' - y$, and $\angle P'PQ = \theta$. Hence, from the right-angled triangle $PP'Q$, we have

$$\left.\begin{array}{c} x' - x = r\cos\theta, \quad y' - y = r\sin\theta, \\ r^2 = (x' - x)^2 + (y' - y)^2, \\ \tan\theta = \dfrac{y' - y}{x' - x}, \end{array}\right\} \ldots\ldots (1);$$

which are the relations required. The two latter are immediately deducible from the two former.

39. It is important to observe that by "*the angle which PP' makes with OX*," we mean the angle through which OX must be turned towards OY, before it becomes parallel to PP', with the point X on the same side of O that P' is of P. Thus, in (figs. 9 and 10), θ is the angle which PP' makes with OX, and ϕ the angle which $P'P$ makes with OX.

Also by the distance PP' we mean the distance measured from P to P', and not from P' to P.

These remarks are made to avoid ambiguity in speaking of angles and distances, and will be found particularly important when we come to consider distances measured along revolving lines.

PROP. II.

40. To determine the area of the trapezium $PMM'P'$ (same figure).

Let A be the required area; then

$$\begin{aligned} 2A &= 2 \text{ area } PMM'Q + 2 \text{ area } PQP' \\ &= 2y(x' - x) + (y' - y)(x' - x) \\ &= (y' + y)(x' - x), \end{aligned}$$

which gives the required area.

PROP. III.

41. To find the area of the triangle whose angular points are (xy), $(x'y')$, $(x''y'')$.

Let $PP'P''$ (fig. 11) be the triangle, draw PM, $P'M'$, $P''M''$ perpendicular to OX; then, if A be the required area, we have

$$A = PMM'P' + P'M'M''P'' - PMM''P''.$$

Hence, by the preceding proposition,

$$2A = (y' + y)(x' - x) + (y'' + y')(x'' - x') - (y'' + y)(x'' - x),$$

$$\text{or } 2A = (y' + y)(x' - x) + (y'' + y')(x'' - x') + (y + y'')(x - x''),$$

which gives the required area in a remarkably symmetrical form.

42. COR. In exactly the same way we may shew that twice the area of any polygon, whose angular points are (xy), $(x'y')$, $(x''y'')\ldots.(x^{(n)}y^{(n)})$, is equal to

$$(y'+y)(x'-x) + (y''+y')(x''-x') + (y'''+y'')(x'''-x'')\ldots + (y+y^{(n)})(x-x^{(n)}).$$

By this formula we may easily determine the area of a field

whose boundaries have been determined by means of what are called offsets in surveying.

Method of representing Curves by Equations.

43. We shall now explain the manner in which co-ordinates may be applied to determine the forms of curves; it is in this application of them that Co-ordinate Geometry chiefly consists.

Let x and y be the co-ordinates of any point P, and suppose that there is given, not x and y, but only some relation between them; then we may assign any value we please to x, and, by means of the given relation between x and y, find a corresponding value of y. If we give to x any set of values OM, OM', OM'', &c. (fig. 12), and find the corresponding values of y, MP, $M'P'$, $M''P''$, &c. suppose, the point we are considering may be in any of the positions P, P', P'', &c.. . . .; and, since the values we give to x may differ from each other as little as we please, it is clear that there is, *in general*, some continuous line or curve in which all the positions of the point will be found. We say, *in general*, because in certain cases the point may occupy only isolated positions, as we shall presently shew. We have supposed that there is only one value of y for each value of x, but the given relation may furnish more than one value; in such a case all that we have just said still holds true, only the curve in which the various positions of P are found will consist of more than one branch, as in (fig. 13).

44. Hence it appears that, when there is given not x and y, but only some relation between x and y, the position of the point (xy) is indeterminate, but is so far restricted as to be always found upon some line or curve (in general), the form of which depends, of course, upon the nature of the given relation.

45. When an indeterminate point is restricted by conditions of any kind to occupy some one of a particular series of positions, that series of positions is called the *locus* of the point. A relation, therefore, of any kind between x and y represents in general some locus, namely, that series of positions which the point (xy) may occupy consistently with the given relation.

46. In the following pages we shall only consider those relations between x and y which may be expressed by ordinary equations (for there are relations which cannot be expressed by means of ordinary equations); and therefore every locus, we shall be concerned with, will be represented by some equation between x and y, by means of which we shall investigate its nature and properties.

47. We therefore define the locus, represented by an equation between x and y, to be the assemblage of all the points whose co-ordinates satisfy that equation.

48. Hence if h and k be quantities which, substituted for x and y, satisfy the equation, (hk) must be a point of the locus.

49. The following examples will shew the nature of this method of representing loci by means of equations.

To determine the nature of the locus represented by the equation $y = x - a$, a being some known positive distance.

Take OM (fig. 14) as any value of x, let $OA = a$, draw MP perpendicular to OX, and take $MP = MA$; then, since $MP = OM - OA$, MP is the value of y corresponding to the value OM of x; and therefore P is a point of the locus to be determined.

Now, since $MP = MA$, the angle $PAM = 45°$; therefore P is a point on the right line drawn through A at an angle 45° to the axis of x; and this is true whatever value of x we suppose OM to be. Therefore all the positions of the point (xy) are found on this line; and hence the equation $(y = x - a)$ represents a right line drawn at an angle 45° to the axis of x, and cutting it at a distance a from the origin.

To determine the nature of the locus represented by the equation, $x^2 - 2ax = 2ay - y^2 - a^2$.

This equation may evidently be put in the form

$$(x - a)^2 + (y - a)^2 = a^2 \ldots (1).$$

Now, by Art. 38, $(x - a)^2 + (y - b)^2$ is the square of the distance of the point (xy) from the point (aa); therefore, by the equation (1), the point (xy) is always at a distance a from the point (aa).

Hence, if C and P (fig. 15) be the points (aa) and (xy), CP is equal to a whatever value we give to x. Therefore the given equation represents a circle whose centre is C and radius a.

Having thus explained the method of co-ordinates, we now proceed to apply it in detail to the investigation of the properties of right lines, circles, and various other curves, and the solution of several interesting and important problems.

CHAPTER III.

OF THE EQUATION OF THE RIGHT LINE. EQUATIONS OF RIGHT LINES SUBJECT TO VARIOUS CONDITIONS. MISCELLANEOUS PROPOSITIONS. PROBLEMS.

Of the General Equation of a Right Line.

PROP. IV.

50. To shew that the general equation of the first degree between x and y represents a right line.*

Every equation of the first degree between x and y is included in the form

$$Ax + By = C \quad \ldots\ldots\ldots\ldots\ldots \quad (1),$$

A, B and C being any quantities independent of x and y; and this equation is therefore called the *general* equation of the first degree.

Let $(x'y')$, $(x''y'')$ be any two points of the locus, whatever it be, represented by (1); then x', y', and x'', y'' put for x and y must satisfy (1) (Art. 47), and we have therefore

$$Ax' + By' = C,$$
$$Ax'' + By'' = C;$$

and, subtracting these equations, we find

$$\frac{A}{B} + \frac{y'' - y'}{x'' - x'} = 0.$$

Now if θ be the angle which the line joining $(x'y')$ and $(x''y'')$ makes with the axis of x, we have, by Prop. I.,

$$\tan\theta = \frac{y'' - y'}{x'' - x'}.$$

Hence we find $\qquad \tan\theta = -\frac{A}{B}.$

* Hence it is that an equation of the first degree is often called a *linear* equation.

It appears therefore that the right line joining any two points of the locus represented by (1) always makes an invariable angle with the axis of x; which can only be true of a right line. Therefore (1) represents a right line. Q. E. D.

51. Cor. Hence $-\frac{A}{B}$ is the tangent of the angle which the right line $(Ax + By = C)$* makes with the axis of x; *i.e.* the line OX must turn through an angle $\tan^{-1} -\left(\frac{A}{B}\right)$ towards OY, in order to become parallel to the line $(Ax + By = C)$ (see Art. 39). Hence, if AB (in either fig. 16 or fig. 17) be the line represented by the equation, BAX, not BAO, is the angle whose tangent is $-\frac{A}{B}$.

Prop. V.

52. To determine, conversely, the general equation of a right line.

Let AB (fig. 18) be any right line, P any point of it, and $OM\,(= x)$, $MP\,(= y)$, the co-ordinates of P: then, whatever be the position of P on AB, we have $PM : MA$ in some invariable ratio, $B : A$ suppose; hence, putting $OA = a$, we find

$$\frac{y}{a - x} = \frac{B}{A}, \text{ or } Ax + By = Ba = C, \text{ suppose};$$

which equation, being a relation between the co-ordinates of any point of the right line, is the equation required.

Prop. VI.

53. To determine the portions which the line $(Ax + By = C)$ cuts off from the co-ordinate axes OX, OY.

Let BA (fig. 19) be the line $(Ax + By = C)$, meeting OX and OY in A and B respectively; then OA, OB are the portions the line cuts off from OX, OY. Now A is a point of the right line, and the co-ordinates of A are OA and 0; therefore the equation must be satisfied when we put $x = OA$, $y = 0$; *i.e.* OA is the value of x got from the equation by putting $y = 0$.

* By the line $(Ax + By = C)$ we mean the line represented by this equation.

Similarly OB is the value of y got by putting $x = 0$. Hence we have

$$OA = \frac{C}{A}, \quad OB = \frac{C}{B}. \quad \text{Q. E. F.}$$

54. Cor. Hence we may easily construct the right line represented by any given equation, as the following examples will shew, viz.:

$$3x + 2y = 6a \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$3x - 2y = 6a \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$-3x + 2y = 6a \ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$3x + 2y = -6a \ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$x = a \ldots\ldots\ldots\ldots\ldots\ldots (5),$$
$$y = b \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

In (1) we have $x = 2a$ when $y = 0$, and $y = 3a$ when $x = 0$; therefore take $OA = 2a$, $OB = 3a$ (fig. 20), and AB is the right line represented by (1).

In (2), $OA = 2a$, $OB = -3a$, therefore fig. 21 represents (2).

In (3), $OA = -2a$, $OB = 3a$, therefore fig. 22 represents (3).

In (4), $OA = -.2a$, $OB = -2b$, therefore fig. 23 represents (4).

(5) represents a series of points all at the same perpendicular distance from OY, *i.e.* a right line parallel to OY (fig. 24): and similarly (6) represents a right line parallel to OX (fig. 25). (5) and (6) may be put in the forms

$$x + 0 \,.\, y = a, \qquad 0 \,.\, x + y = a,$$

from which it appears that $OA = a$, $OB = \infty$ in the former, and $OA = \infty$, $OB = a$ in the latter.

55. There is one case to which this method cannot be applied, namely, when the equation occurs in the form

$$Ax + By = 0,$$

for in this case OA and OB are each zero, which shews that the line passes through the origin, but does not determine its position. But we may immediately find its position, since, by Prop. IV. Cor., the tangent of the angle (θ) it makes with OX is $-\dfrac{A}{B}$. For example, to construct the lines represented by

$$x + y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$
$$x - 3y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

In (7), $\tan\theta = -1$, and therefore $\theta = \frac{3\pi}{4}$, and fig. 26 represents the line.

In (8), $\tan\theta = \frac{1}{3}$, and therefore if we take $OM = 3$, $MP = 1$, (fig 27), the line drawn through O and P is that represented by (8).

56. Of certain forms in which the equation of a right line may be put.

(1) If we divide the equation by C, and put $\frac{A}{C} = \frac{1}{a}, \frac{B}{C} = \frac{1}{b}$, it assumes the form $$\frac{x}{a} + \frac{y}{b} = 1.*$$

Here a and b are the portions which the line cuts off from the axes OX, OY; for $x = a$ when $y = 0$, and $y = b$ when $x = 0$.

(2) If we divide by B and put $-\frac{A}{B} = m$, $\frac{C}{B} = c$, the equation assumes the form $$y = mx + c.$$

Here m is the tangent of the angle which the line makes with the axis of x; and, since $y = c$ when $x = 0$, c is the portion cut off from the axis of y. If therefore we take $OB = c$, and draw the line AB making the angle $BAX = \tan^{-1}m$ (as in fig. (28) if m be positive, or as in fig. (29) if m be negative), AB is the line represented by the equation.

(3) If we put $\tan\theta$ for m in the equation $y = mx + c$, multiply by $\cos\theta$, and put $c\cos\theta = p$, it assumes the form

$$y\cos\theta - x\sin\theta = p.$$

Here θ is the angle which the line makes with the axis of x; and, if we draw OQ perpendicular to AB, $p = OQ$; for $p = c\cos\theta = OB\cos BOQ = OQ$.

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* This equation may be proved geometrically as follows. In fig. (18) we have
$$\frac{NP}{OA} = \frac{BP}{BA}, \quad \frac{MP}{OB} = \frac{AP}{AB};$$
$$\therefore \frac{NP}{OA} + \frac{MP}{OB} = \frac{AB}{AB} = 1,$$
$$\text{or } \frac{x}{a} + \frac{y}{b} = 1.$$

57. Hence it appears that, when the equation of a right line is put in the form

$$\frac{x}{a} + \frac{y}{b} = 1,$$

a and b are the portions it cuts off from OX and OY.

When it is put in the form

$$y = mx + c,$$

m is the tangent of the angle it makes with OX, and c is the portion it cuts off from OY.

And when it is put in the form

$$y \cos \theta - x \sin \theta = p,$$

θ is the angle it makes with OX, and p is the perpendicular upon it from O.

Of the Equations of Right Lines subject to various Conditions.

58. When we have occasion to write down the equations of several different lines at the same time, we shall not make any distinction between the x's and y's in the different equations, but write the same x and y in each of them: which amounts to supposing that (xy) is any point on *any* of the lines. Thus, suppose we have to consider two lines; then, instead of saying, let their equations be

$$Ax + By = C, \quad A'x' + B'y' = C',$$

we shall simply say, let their equations be

$$Ax + By = C \ldots\ldots(1), \quad A'x + B'y = C' \ldots\ldots(2).$$

This being the case, it will be important to remember that (xy) in (1) denotes a point on the line represented by (1), and (xy) in (2) a point on the line represented by (2); and that therefore (xy) in (1) must be a different point from (xy) in (2), *except* at the intersection of the two lines.

59. The general co-ordinates of any point of a locus, between which the equation of that locus is a relation, have been called by some writers the *current co-ordinates* of the locus; and the other quantities involved in the equation, which are independent of these co-ordinates, and serve to determine the position and form of the locus, are generally called the *parameters* of the locus. Thus, in the equation of the right line,

x and y are the current co-ordinates, and A, B, C the parameters. These names are very convenient in many cases.

Prop. VII.

60. To determine the form of the equation of a right line when it is restricted to pass through a point (hk), or through two points (hk) and $(h'k')$.

Let the equation of the line be

$$Ax + By = C \quad \ldots\ldots\ldots\ldots (1).$$

Then, since (hk) is a point of it, we have

$$Ah + Bk = C \quad \ldots\ldots\ldots\ldots (2).$$

(2) determines one of the parameters A, B, C in terms of the other two, and expresses the condition necessary in order that (1) may pass through (hk). If we subtract (2) from (1), in order to get rid of C, we find

$$A(x-h) + B(y-k) = 0 \quad \ldots\ldots\ldots (3);$$

and, since this equation is manifestly satisfied when $x = h$ and $y = k$, it is the general equation of a right line restricted to pass through the point (hk).

If the line also pass through the point $(h'k')$, (3) must be satisfied when $x = h'$ and $y = k'$; therefore

$$A(h'-h) + B(k'-k) = 0 \quad \ldots\ldots\ldots (4),$$

(4) is the condition necessary in order that the line (3) may pass through $(h'k')$; and, if we substitute in (3) the value of $A \div B$ got from (4), (3) becomes

$$\frac{x-h}{h'-h} - \frac{y-k}{k'-k} = 0 \quad \ldots\ldots\ldots\ldots (5),$$

which is the equation of a right line passing through the two points (hk) and $(h'k')$.

61. Cor. Hence the equation of the line drawn through (hk) at an angle θ to the axis of x is

$$(x-h)\sin\theta - (y-k)\cos\theta = 0,$$

as is evident by putting in (3) for $\frac{B}{A}$ its value $(-\tan\theta)$.

Prop. VIII.

62. To find the conditions necessary in order that two right lines $(Ax + By = C)$ and $(A'x + B'y = C')$ may be parallel or perpendicular to each other.

If θ and θ' be the angles which these lines make with OX, we have, by Prop. IV. Cor.,

$$\tan\theta = -\frac{A}{B}, \quad \tan\theta' = -\frac{A'}{B'}.$$

Hence, if the lines be parallel and therefore $\theta' = \theta$, we have

$$\frac{A'}{B'} = \frac{A}{B} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

And if the lines be at right angles, and therefore $\theta' = \theta + \frac{\pi}{2}$ which makes $\tan\theta' = -\cot\theta$, we have

$$\frac{A'}{B'} = -\frac{B}{A} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

(1) and (2) are the conditions required.

63. Cor. 1. Hence the lines $(Ax + By = C)$ and $(Ax + By = C')$ are parallel, and the lines $(Ax + By = C)$ and $(Bx - Ay = C')$ are perpendicular to each other.

Cor. 2. Hence, and by Prop. VII., the equation of the lines drawn parallel and perpendicular to $(Ax + By = C)$, through the point (hk), are respectively,

$$A(x - h) + B(y - k) = 0,$$
$$B(x - h) - A(y - k) = 0.$$

Miscellaneous Propositions respecting Right Lines.

Prop. IX.

64. To determine the point of intersection of two right lines, viz.

$$Ax + By = C \quad\ldots\ldots\ldots\ldots\ldots (1),$$
$$A'x + B'y = C' \quad\ldots\ldots\ldots\ldots\ldots (2).$$

As we have explained in Art. 58, x and y are not the same quantities in (1) that they are in (2), except they belong to a point common to both the lines: suppose therefore that x and y are the same in (1) and (2), and then (xy) is a point common to the two lines. Now, on this supposition, (1) B' - (2) B and (1) A' - (2) A give

$$x = \frac{CB' - C'B}{AB' - A'B}, \quad y = \frac{CA' - C'A}{BA' - B'A}.$$

The values of x and y thus determined are the co-ordinates of the point of intersection required.

65. Cor. These values of x and y become infinite when $AB' - A'B = 0$, as they ought to do, for then, by Art. 62, the two lines are parallel.

Prop. X.

66. To find the angle which the line $(A'x + B'y = C')$ makes with the line $(Ax + By = C)$.

Let ϕ be the required angle, and θ', θ the angles which the two lines make with the axis of x; then we have

$$\phi = \theta' - \theta, \quad \tan\theta' = -\frac{A'}{B'}, \quad \tan\theta = -\frac{A}{B};$$

and therefore
$$\tan\phi = \frac{-\dfrac{A'}{B'} + \dfrac{A}{B}}{1 + \dfrac{A'}{B'}\dfrac{A}{B}};$$

or
$$\tan\phi = \frac{AB' - A'B}{AA' + BB'},$$

which determines ϕ.

Cor. Hence $\cos\phi = \dfrac{AA' + BB'}{\sqrt{\{(A^2 + B^2)(A'^2 + B'^2)\}}}$.

Prop. XI.

67. To find the equation of a line making a given angle (ϕ) with the line $(Ax + By = C)$.

The angle which the required line makes with the axis of x is $\phi + \tan^{-1}\left(-\dfrac{A}{B}\right)$, and its tangent is therefore

$$\frac{\tan\phi - \dfrac{A}{B}}{1 + \tan\phi\,\dfrac{A}{B}}, \quad \text{or} \quad \frac{B\sin\phi - A\cos\phi}{B\cos\phi + A\sin\phi}.$$

Hence the equation of the required line is, by Art. 51,

$$(A\cos\phi - B\sin\phi)\,x + (A\sin\phi + B\cos\phi)\,y = C'.$$

Prop. XII.

68. To find the length (r) of a right line drawn, at an angle θ to OX, from a point (hk) to a line $(Ax + By = C)$.

Let us suppose (xy) to be the point where the former line

meets the latter; then, by Prop. I.,

$$x = h + r\cos\theta, \quad y = k + r\sin\theta;$$

and, substituting these values in $Ax + By = C$, we have

$$r(A\cos\theta + B\sin\theta) + Ah + Bk = C;$$

and therefore
$$r = \frac{C - Ah - Bk}{A\cos\theta + B\sin\theta},$$
which is the required distance.

Prop. XIII.

69. To find the perpendicular distance of a point (hk) from a right line $(Ax + By = C)$.

Proceeding as in the last proposition, we have

$$r(A\cos\theta + B\sin\theta) = C - Ah - Bk \;\ldots\ldots\; (1).$$

And since r is perpendicular to $(Ax + By = C)$, we have, by Art. 51,
$$\tan\left(\theta - \frac{\pi}{2}\right) = -\frac{A}{B}, \text{ or } \cot\theta = \frac{B}{A};$$
and therefore
$$r(B\cos\theta - A\sin\theta) = 0 \;\ldots\ldots\ldots\ldots\; (2).$$

Squaring and adding (1) and (2), we find immediately

$$r = \frac{C - Ah - Bk}{\sqrt{(A^2 + B^2)}},\text{*}$$

which is the required perpendicular distance.

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* This result may be arrived at geometrically as follows:

Let AB (fig. 51) be the line $(Ax + By = C)$, and P the point (hk); draw $A'B'$ through P parallel to AB, and draw OQQ' perpendicular to both these lines; then QQ' is evidently the perpendicular distance required. Now the equations of AB and $A'B'$ are

$$Ax + By = C,$$
$$Ax + By = C', \text{ where } C' = Ah + Bk;$$

and we have therefore

$$OQ = OA\sin QAO = \frac{OA}{\sqrt{(1 + \cot^2 QAO)}};$$

which, since $OA = \dfrac{C}{A}$, $\tan QAO = \dfrac{A}{B}$, gives

$$OQ = \frac{C}{\sqrt{(A^2 + B^2)}};$$

and similarly, we find
$$OQ' = \frac{C'}{\sqrt{(A^2 + B^2)}}.$$

Hence
$$QQ' = \frac{C - C'}{\sqrt{(A^2 + B^2)}} = \frac{C - Ah - Bk}{\sqrt{(A^2 + B^2)}}.$$

70. Cor. Hence the perpendicular distance of the line $(Ax + By = C)$ from the origin is $\dfrac{C}{\sqrt{(A^2 + B^2)}}$.

Prop. XIV.

71. To find the equation of a right line passing through the point of intersection of the two lines $(Ax + By = C)$......(1) and $(A'x + B'y = C')$.... (2).

We may put the equation of any right line whatever in the form $P(Ax + By - C) + Q(A'x + B'y - C') = R$ .... (3), for we may make this equation agree with any proposed equation of the first degree, by giving proper values to the disposable quantities P, Q and R.

Now, if (1), (2), and (3) intersect in the same point, they must be satisfied by the same values of x and y; hence, substituting the values of x and y which satisfy (1) and (2) in (3), we find
$$0 = R.$$
Hence (3) becomes
$$P(Ax + By - C) + Q(A'x + B'y - C') = 0 \text{ .... (4),}$$
which is the required equation, and in which P and Q are arbitrary. Indeed it is manifest immediately that (4) represents any line passing through the intersection of (1) and (2); for (4) is evidently satisfied by the values of x and y which satisfy (1) and (2); *i.e.* the intersection of (1) and (2) is a point on (4).

Various Problems respecting Right Lines.

72. To determine the area of the triangle BPB' (fig. 30), having given the equations of the lines BP and $B'P$.

Let the given equations be
$$y = mx + c \text{ ...... (1),} \quad y = m'x + c' \text{ ...... (2),}$$
draw PQ perpendicular to OY, and let A be the area required: then
$$2A = BB'.PQ, \text{ and } BB' = OB - OB'.$$
Hence, since OB and OB' are the values of y got from (1) and (2) by putting $x = 0$, and PQ the value of x got by subtracting (2) from (1), we have
$$2A = \frac{(c' - c)^2}{m' - m},$$

73. To determine the area of a triangle, having given the equations of its three sides.

Let BP, $B'P'$, $B''P''$ (fig. 31) be the three sides of the triangle $PP'P''$, produced to meet OY in B, B', B''; and let the given equations of BP, $B'P'$, $B''P''$ be

$$y = mx + c, \qquad y = m'x + c', \qquad y = m''x + c''.$$

Then, if A be the area required, we have

$$A = BPB' + B'P'B'' - BP''B'';$$

and therefore by the previous problem

$$2A = \frac{(c' - c)^2}{m' - m} + \frac{(c'' - c')^2}{m'' - m'} + \frac{(c - c'')^2}{m - m''}.$$

74. To inscribe a square ($PP'M'M$) (fig. 32) in a triangle BAA', one side of the square being upon the side AA' of the triangle.

Take the side AA' produced and the perpendicular upon it BO as co-ordinate axes; let $OA = a$, $OA' = a'$, $OB = b$; then the equations of AB and $A'B$ are

$$\frac{x}{a} + \frac{y}{b} = 1 \;\ldots\ldots (1), \qquad \frac{x}{a'} + \frac{y}{b} = 1 \;\ldots\ldots\ldots\ldots (2);$$

and, if $PM = c$, the equation of PP' is

$$y = c \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now, since $MPP'M'$ is a square,

$$c = MM' = OM' - OM.$$

Hence, since OM is the value of x got from (1) and (3), and OM' the value got from (2) and (3), we have

$$c = a'\left(1 - \frac{c}{b}\right) - a\left(1 - \frac{c}{b}\right),$$

which gives

$$c = \frac{(a' - a) . b}{a' - a + b},$$

which determines the side of the square required.

75. This suggests the following geometrical construction.

Take $OQ = AA'$, join QA, and draw OP parallel to QA; then P is one corner of the square. For, drawing PP' parallel to AA', we have

$$\frac{PP'}{AA'} = \frac{BP}{BA} = \frac{BO}{BQ},$$

or
$$\frac{PP'}{a' - a} = \frac{b}{a' - a + b},$$
which shews that $PP' = c$.

76. B (fig. 33) is a fixed point on OY, Q is a moveable point on OX, BQP is a right angle, and $BQ : QP$ is an invariable ratio; to determine the locus of P.

Let $OB = b$, $QP : BQ = m : 1$, $OM = x$, and $MP = y$; then, since the triangles OBQ and QPM are similar, we have
$$OQ = \frac{y}{m}, \quad QM = mb, \quad \text{and therefore } x = \frac{y}{m} + mb,$$
or
$$y = m(x - mb).$$
Hence the locus is a right line, which may be constructed thus: take $OR = mOB$, draw RP perpendicular to BR, and RP is the locus required.

77. To determine the equation of the right line passing through one angle of a triangle and cutting the opposite side in a given ratio, having given the equations of the sides of the triangle to be
$$y \cos\theta - x \sin\theta = p \; \ldots\ldots\ldots\ldots \; (1),$$
$$y \cos\theta' - x \sin\theta' = p' \; \ldots\ldots\ldots\ldots \; (2),$$
$$y \cos\theta'' - x \sin\theta'' = p'' \; \ldots\ldots\ldots\ldots \; (3).$$

Let the required line pass through the angle made by (2) and (3), and cut the side (1) in the ratio $m : n$; and let r' and r'' be the distances, drawn parallel to (1) (*i.e.* at an angle θ to OX), of any point (xy) of the required line from (2) and (3). Then, by Prop. XII., we have, (putting $x + r' \cos\theta$, $y + r' \sin\theta$ for x and y, in (2),)
$$r' = \frac{u'}{\sin(\theta - \theta')} \text{ where, for brevity, } u' = p' - y\cos\theta' + x \sin\theta';$$
and similarly
$$r'' = \frac{u''}{\sin(\theta - \theta'')} \text{ where } u'' = p'' - y \cos\theta'' + x \sin\theta'',$$
and we evidently have $r' : r'' :: m : n$; hence
$$\frac{mu''}{\sin(\theta - \theta'')} = \frac{nu'}{\sin(\theta - \theta')} \; \ldots\ldots\ldots\ldots \; (4),$$

or $\dfrac{m}{\sin(\theta-\theta'')}(p''-y\sin\theta''+x\cos\theta'')=\dfrac{n}{\sin(\theta-\theta')}(p'-y\sin\theta'+x\cos\theta')$,
which is the required equation.

78. If three lines, drawn through the three angles of a triangle, and cutting the opposite sides in the ratios $m:n$, $m':n'$, $m'':n''$ respectively, meet in the same point; then

$$mm'm''=nn'n''.$$

For (4) represents one of these lines, and the two others are in like manner represented by

$$\frac{m'u}{\sin(\theta'-\theta)}=\frac{n'u''}{\sin(\theta'-\theta'')}\ \ldots\ldots\ldots\ldots\ (5),$$

$$\frac{m''u'}{\sin(\theta''-\theta')}=\frac{n''u}{\sin(\theta''-\theta)}\ \ldots\ldots\ldots\ldots\ (6),$$

where u (in the same manner as u' and u'') represents

$$(p-y\cos\theta+x\sin\theta).$$

Now, if the three lines meet in the same point, we may suppose x and y, and therefore u, u', u'' to have the same values in the three equations; and then, multiplying the three equations together, we find immediately

$$mm'm''=nn'n''.$$

79. If $m=n$ and $m'=n'$, then this result shews that $m''=n''$. Hence it follows that if two of the lines bisect two sides of the triangle the third will bisect the remaining side.

80. Representing the three sides of the triangle by the same equations (1), (2), and (3), to determine the equation of the right line drawn through the angle formed by (2) and (3) and perpendicular to (1).

By Prop. XIV. the equation of any line passing through the intersection of (2) and (3) is

$$P(y\cos\theta'-x\sin\theta'-p')+Q(y\cos\theta''-x\sin\theta''-p'')=0\ \ldots\ (7);$$

and, this line being perpendicular to (1), we have

$$-\frac{P\cos\theta'+Q\cos\theta''}{P\sin\theta'+Q\sin\theta''}=\tan\theta,$$

or

$$P\cos(\theta-\theta')+Q\cos(\theta-\theta'')=0.$$

Hence (7) becomes

$$\frac{u'}{\cos(\theta-\theta')}=\frac{u''}{\cos(\theta-\theta'')}\ \ldots\ldots\ldots\ldots\ (8).$$

81. The equations of the other two perpendiculars are in like manner

$$\frac{u''}{\cos(\theta' - \theta'')} = \frac{u}{\cos(\theta' - \theta)} \cdots\cdots\cdots\cdots (9),$$

$$\frac{u}{\cos(\theta'' - \theta)} = \frac{u'}{\cos(\theta'' - \theta')} \cdots\cdots\cdots (10).$$

Now if we suppose x and y, and therefore u'', to have the same values in (8) and (9), and if we multiply the two equations together, we find

$$\frac{u'}{\cos(\theta' - \theta'')} = \frac{u}{\cos(\theta - \theta'')},$$

which is identical with (10). Hence the same values of x and y which satisfy (8) and (9) satisfy (10) also; *i.e.* the intersection of (8) and (9) is a point on (10). It appears therefore that the three perpendiculars meet in the same point.

82. To determine the equation of the right line drawn through and bisecting the angle made by (2) and (3).

As before, (7) represents the line drawn through the angle formed by (2) and (3); but since it bisects the angle formed by two lines inclined at angles θ', θ'' to OX, it must make an angle $\dfrac{\theta' + \theta''}{2}$ with OX. We have therefore

$$\frac{P \sin\theta' + Q \sin\theta''}{P \cos\theta' + Q \cos\theta''} = \tan\frac{\theta' + \theta''}{2},$$

or
$$P \sin\frac{\theta'' - \theta'}{2} + Q \sin\frac{\theta' - \theta''}{2};$$

and therefore
$$P = Q.$$

Hence the equation required is

$$u' = u'' \cdots\cdots\cdots\cdots\cdots\cdots (11).$$

83. The equations of the bisectors of the other two angles are in like manner

$$u'' = u \cdots\cdots\cdots\cdots\cdots\cdots (12),$$

$$u = u' \cdots\cdots\cdots\cdots\cdots\cdots (13).$$

And reasoning exactly as in the previous case, we may immediately shew that the three bisectors meet in the same point.

It is remarkable that the equation of the line passing through

and bisecting the angle formed by the lines ($y\cos\theta - x\sin\theta = p$) and ($y\cos\theta' - x\sin\theta' = p'$) should be

$$y\cos\theta - x\sin\theta - p = y\cos\theta' - x\sin\theta' - p'.$$

84. Having given a set of n points (hk), $(h'k')$, $(h''k'')$, &c., to draw a right line in such a position that the sum of the perpendiculars let fall upon it from the given points shall be equal to zero, (*i. e.* the sum of those on one side equal to the sum of those on the other).

Let the equation of the required line be

$$y\cos\theta - x\sin\theta = p \;\ldots\ldots\ldots\ldots\; (1),$$

then the perpendiculars upon it from the given points are, by Prop. XIII.,

$$p - k\cos\theta + h\sin\theta,\quad p - k'\cos\theta + h'\sin\theta,\quad p - k''\cos\theta + h''\sin\theta,\ \&c.\ \ldots\ldots$$

Hence, if the sum of the n perpendiculars be zero, we have

$$np - (k + k' + k''\ldots)\cos\theta + (h + h' + h''\ldots)\sin\theta = 0;$$

or, for brevity, putting

$$k + k' + k''\ldots\ldots = nk_1,\quad h + h' + h''\ldots\ldots = nh_1,$$

we have $$p = k_1\cos\theta - h_1\sin\theta;$$

and, putting this value of p in (1), we have

$$(y - k_1)\cos\theta - (x - h_1)\sin\theta = 0,$$

which is the required equation. It is the equation of *any* line passing through the point (h_1k_1), for θ may have any value whatever.

Hence, if we draw any right line through the point whose co-ordinates are $\dfrac{h + h' + h'' + \&c.}{n}$ and $\dfrac{k + k' + k'' + \&c.}{n}$, the sum of the perpendiculars let fall upon it from the points (hk), $(h'k')$, $(h''k'')$, &c. is zero.

CHAPTER IV.

OF OBLIQUE CO-ORDINATES. MODIFICATIONS NECESSARY IN THE PREVIOUS RESULTS WHEN THE CO-ORDINATES ARE SUPPOSED TO BE OBLIQUE. POLAR CO-ORDINATES. TRANSFORMATION OF CO-ORDINATES.

Of Oblique Co-ordinates. How the previous results are modified when the Co-ordinates are Oblique.

85. It is sometimes convenient to make use of axes of co-ordinates inclined to each other at some given angle different from a right angle. In such a case, if OY, OX be the axes (fig. 34), and PM, PN respectively parallel to them, we consider OM and ON to be the co-ordinates of the point P. Axes and co-ordinates of this kind are denominated *oblique* axes and *oblique* co-ordinates; those we have been previously making use of being called *rectangular* in contradistinction.

86. All that we have said, as far as the end of Art. 37, is equally true whether the co-ordinates be rectangular or oblique, and the same may be said of Articles 43–48, 53–55, the first form of 56, 58–60, 64, and 71. In the other Articles some alteration must be made.

87. Thus, instead of the formulæ in Prop. I., we have the following (ω being the angle YOX, or, as it is called, the angle of ordination), viz.

$$x' - x = r\,\frac{\sin(\omega - \theta)}{\sin \omega}, \quad y' - y = r\,\frac{\sin \theta}{\sin \omega},$$

$$r^2 = (x' - x)^2 + (y' - y)^2 + 2\,(x' - x)\,(y' - y)\cos \omega,$$

$$\frac{\sin \theta}{\sin(\omega - \theta)} = \frac{y' - y}{x' - x}.$$

The truth of these formulæ is evident from fig. 35 (which corresponds to fig. 8), in which

$$PP' = r,\ PQ = x' - x,\ P'Q = y' - y,\ \angle P'PQ = \theta,\ \angle P'QP = \pi - \omega.$$

88. Prop. IV. is equally true whether the co-ordinates be rectangular or oblique; only, in the latter case, instead of saying that

$$\tan\theta = \frac{y' - y}{x' - x} = -\frac{A}{B},$$

we must say, that, by Art. 87,

$$\frac{\sin\theta}{\sin(\omega - \theta)} = \frac{y' - y}{x' - x} = -\frac{A}{B}.$$

89. Hence, instead of the Cor. to Prop. IV., we must say, that,

$$A\sin(\omega - \theta) + B\sin\theta = 0;$$

therefore $\quad A(\sin\omega - \cos\omega\tan\theta) + B\tan\theta = 0;$

and therefore $\quad \tan\theta = \dfrac{A\sin\omega}{A\cos\omega - B},$

which is the tangent of the angle the line $(Ax + By = C)$ makes with the axis of x when the co-ordinates are oblique.

90. The condition of parallelism in Prop. VIII. is equally true whether the co-ordinates be rectangular or oblique; only in the latter case, instead of saying that

$$\tan\theta = -\frac{A}{B},\quad \tan\theta' = -\frac{A'}{B'},$$

we must say, that, by Art. 87,

$$\frac{\sin\theta}{\sin(\omega - \theta)} = -\frac{A}{B},\quad \frac{\sin\theta'}{\sin(\omega - \theta')} = -\frac{A'}{B'}.$$

91. The condition of perpendicularity in Prop. VIII. must be altered as follows.

By Art. 89, we have

$$\tan\theta\,.(A\cos\omega - B) = A\sin\omega,$$
$$\tan\theta\,.(A'\cos\omega - B') = A'\sin\omega;$$

and hence, since $\quad \tan\theta.\tan\theta' = -1, \quad$ we have

$$(A\cos\omega - B)(A'\cos\omega - B') + AA'\sin^2\omega;$$

or $\quad A'(A - B\cos\omega) + B'(B - A\cos\omega) = 0,$

which is the required condition instead of (2), Prop. VIII.

Hence it follows that

$$-(B - A\cos\omega)\,x + (A - B\cos\omega)\,y = 0$$

is the equation of a line perpendicular to the line $(Ax + By = C)$, the co-ordinates being oblique.

92. Prop. x. must be altered as follows.

By Art. 89, we have

$$\tan\phi = \frac{\dfrac{A'\sin\omega}{A'\cos\omega - B'} - \dfrac{A\sin\omega}{A\cos\omega - B}}{1 + \dfrac{A'\sin\omega}{A'\cos\omega - B'}\cdot\dfrac{A\sin\omega}{A\cos\omega - B}}$$

$$= \frac{(AB' - A'B)\sin\omega}{AA' + BB' - (A'B + B'A)\cos\omega}.$$

From this expression, by putting $\phi = 0$, or $\frac{\pi}{2}$, we obtain the conditions necessary in order that two right lines may be parallel or perpendicular to each other.

93. Lastly, Props. XII. and XIII. must be altered as follows.

By Art. 87, we must substitute in the equation $Ax + By = C$ the values

$$x = h + r\,\frac{\sin(\omega - \theta)}{\sin\omega}, \quad y = k + r\,\frac{\sin\theta}{\sin\omega};$$

and then we have

$$r\,\{A\sin(\omega - \theta) + B\sin\theta\} = (C - Ah - Bk)\sin\omega \quad \ldots\ldots \quad (1),$$

which gives r.

If r be perpendicular to $(Ax + By = C)$, $\theta - \frac{\pi}{2}$ must be the angle which the latter line makes with the axis of x; and therefore, by Art. 89, we have

$$\frac{A\sin\omega}{A\cos\omega - B} = \tan\left(\theta - \frac{\pi}{2}\right) = -\cot\theta \;\ldots\ldots\ldots \; (2).$$

If for a moment we put $B - A\cos\omega = \lambda$, $A\sin\omega = \mu$, (1) and (2) $\times\, r$ become

$$r\,(\mu\cos\theta + \lambda\sin\theta) = (C - Ah - Bk)\sin\omega,$$

$$r\,(\mu\sin\theta - \lambda\cos\theta) = 0;$$

hence, squaring and adding, we find

$$r^2(\lambda^2 + \mu^2) = (C - Ah - Bk)^2 \sin^2 \omega ;$$

and therefore $$r = \frac{(C - Ah - Bk) \sin \omega}{\sqrt{(A^2 - 2AB \cos \omega + B^2)}};$$

which is the expression for the perpendicular distance of (hk) from $(Ax + By = C)$, the co-ordinates being oblique.

Of Polar Co-ordinates.

94. Besides the method of rectangular and oblique co-ordinates just explained, there is another method often employed, namely, that of *polar* co-ordinates, which consists in representing the position of a point in the following manner.

Let P (fig. 36) be the point whose position we wish to represent; choose any line OA and any point O upon it, join OP, assume $OP = r$ and $\angle POA = \theta$; then the position of the point P is given when the values of the quantities r and θ are given. These quantities are called the *polar* co-ordinates of P; O is called the *pole*, r the *radius vector*, and OA the *prime radius*; θ is sometimes called the *vectorial angle*, though it is not usually distinguished by any particular name.

95. The sign $-$ is applied to polar co-ordinates on exactly the same principles as those already explained in the case of rectangular co-ordinates. Thus, if r represent any distance measured from O towards P, $-r$ represents an equal distance measured in the opposite direction from O towards P'; and if θ represent any angle measured from A towards P, $-\theta$ will represent an equal angle measured in the opposite direction from A towards Q. A few examples will make this clearer.

96. Let a be any distance measured from O towards B, θ being the angle which OP makes with OA (see Art. 39); then

$\theta = \frac{\pi}{4}$ $r = a$ represents P in fig. 37,

$\theta = \frac{\pi}{2}$ $r = -a$ .... fig. 38,

$\theta = \frac{3\pi}{4}$ $r = -a$ .... fig. 39,

$\theta = \pi$ $r = -a$ .... fig. 40,

$\theta = \frac{7\pi}{4}$ $r = a$ .... fig. 41.

It is important to observe that the direction in which r is measured depends, not only upon its sign, but also on the value of θ; thus when $\theta = \frac{3\pi}{4}$ and $r = -a$, r must be measured from O towards P (fig. 39); and when $\theta = \frac{7\pi}{4}$ and $r = a$, r must be measured in exactly the same direction. Again, when $\theta = 0$ and $r = a$, and when $\theta = \pi$ and $r = -a$, r must be measured in both cases from O towards A.

Prop. XV.

97. To determine the distance between two points whose polar co-ordinates are (r, θ), (r', θ').

Let P, P' (fig. 42) be the two points, $POA = \theta$, $P'OA = \theta'$, $OP = r$, $OP' = r'$; then in the triangle POP', we have

$$PP'^2 = OP^2 + OP'^2 - 2OP.OP' \cos P'OP,$$

or

$$PP' = \sqrt{(r^2 + r'^2 - 2rr' \cos(\theta' - \theta))},$$

which is the required distance.

Prop. XVI.

98. To determine the polar equation of a right line.

In the same manner that curves and lines are represented by equations between x and y, they may also be by equations between r and θ, which are called *polar* equations. Thus, let BP (fig. 43) be any right line, P any point of it, OP $(= r)$, and POA $(= \theta)$, the polar co-ordinates of P; and let $OB = a$, $PBA = \beta$; then

$$OP : OB :: \sin PBA : \sin BPO,$$

or

$$\frac{r}{a} = \frac{\sin \beta}{\sin(\beta - \theta)},$$

which is a relation between the polar co-ordinates of any point of the right line, and is therefore the polar equation required.

99. Cor. This equation may be put in the form,

$$Ar \cos \theta + Br \sin \theta = C,$$

which is therefore the general form of the polar equation of a right line.

Transformation of Co-ordinates.

100. It is often necessary to change, or, as it is called, to transform the co-ordinates which represent the position of any point from one set to another; for example, supposing the position of the point expressed by rectangular co-ordinates, to express it by oblique or polar co-ordinates. We now proceed to shew how transformation of this kind may be effected in various cases.

PROP. XVII.

101. To transform polar co-ordinates into rectangular, or *vice versâ.*

Let P (fig. 43 *bis.*) be any point, and $OP\ (=r)$, $POA\ (=\theta)$ its polar co-ordinates; take OAX as axis of x, and OY drawn at right angles to OA as axis of y; draw PM perpendicular to OX, and then $OM\ (=x)$, $MP\ (=y)$ will be the rectangular co-ordinates of P. Hence we have

$$x = r\cos\theta,\quad y = r\sin\theta \,\ldots\ldots\ldots\, (1),$$

which give x and y in terms of r and θ; also

$$r = \sqrt{(x^2+y^2)},\quad \tan\theta = \frac{y}{x} \,\ldots\ldots\ldots\, (2),$$

which give r and θ in terms of x and y. And thus, from the polar we may obtain the rectangular co-ordinates of a point, or *vice versâ.*

Hence, by making the substitutions (1) in any equation between x and y, we change it into a polar equation; and by making the substitutions (2) in an equation between r and θ, we change it into a rectangular equation.

For example, by putting $x = r\cos\theta$, $y = r\sin\theta$, in the equation $Ax + By = C$, it becomes $Ar\cos\theta + Br\sin\theta = C$; which agrees with the result of Art. 99.

PROP. XVIII.

102. To turn rectangular axes round their origin through any required angle (a).

Let x and y be the rectangular co-ordinates of any point P (fig. 44), and r and θ its polar co-ordinates, as in the last article; then

$$x = r\cos\theta = r\cos(\theta - a + a),$$
$$= r\cos(\theta - a)\,.\cos a - r\sin(\theta - a)\,.\sin a,$$

and $$y = r\sin\theta = r\sin(\theta - a)\,.\cos a + r\cos(\theta - a)\,.\sin a.$$

Now $r\cos(\theta-a)$ and $r\sin(\theta-a)$ are evidently the co-ordinates of the point referred to the rectangular axes OX', OY', $\angle X'OX$ being equal to a; call these co-ordinates x' and y'; then

$$\left.\begin{aligned} x &= x'\cos a - y'\sin a, \\ y &= y'\cos a + x'\sin a, \end{aligned}\right\} \ldots\ldots (1).$$

Hence, if we have any equation between x and y, and we make in it the substitutions (1), it becomes an equation between x' and y'. This process, therefore, turns the axes through an angle a; *i.e.* it introduces co-ordinates referred to the axes OX', OY', in place of those referred to OX, OY.

103. For example, let the equation between x and y be

$$x^2 - y^2 = a^2 \ldots\ldots\ldots\ldots (2),$$

and let $a = 45°$; then the equations (1) become

$$x = \frac{1}{\sqrt{2}}(x'-y'), \quad y = \frac{1}{\sqrt{2}}(x'+y');$$

hence (2) becomes

$$(x'-y')^2 - (x'+y')^2 = 2a^2,$$

or

$$2x'y' + a^2 = 0 \ldots\ldots\ldots (3).$$

Thus the equation (2) referred to OX, OY, is equivalent to the equation (3) referred to OX', OY', $\angle X'OX$ being 45°.

104. In making the substitutions (1) we may omit the dashes for the same reasons as those explained in Art. 58, and then we may enunciate the result of this Article in the following manner. *To turn the axes of co-ordinates through any angle* (a), we have only to put $x\cos a - y\sin a$ in place of x, and $y\cos a + x\sin a$ in place of y; remembering that x and y, after these substitutions, are referred to OX' and OY'.

Prop. XIX.

105. To transfer the origin to a point (hk) without altering the direction of the axes.

Let $O'X'$, $QO'Y'$ be the new axes (fig. 45) which are respectively parallel to OX and OY, and draw $PM'M$ parallel to OY; then $OM = OQ + O'M'$, $MP = QO' + M'P$; or, since O' is (hk),

$$x = h + x', \quad y = k + y';$$

which substitutions, made in an equation between x and y, change it into an equation between x' and y'.

Hence, suppressing the dashes, as in the last Article, it appears that we transfer the origin to the point (hk), without altering the directions of the axes, by putting $h + x$ in place of x and $k + y$ in place of y.

106. Cor. 1. Hence, to transfer the origin to a point (hk), and to turn the axes through an angle a, we have only to put $h + x \cos a - y \sin a$ in place of x, and $k + y \cos a + x \sin a$ in place of y.

107. Cor. 2. In the case of polar co-ordinates it appears, in exactly the same way, that to turn the prime radius through any angle (a), we have only to put $\theta + a$ for θ.

Prop. XX.

108. To reverse the positive direction of the axis of x, or of the axis of y.

If we suppose OX' (fig. 2) to become the positive axis of x instead of OX, every distance measured along $X'X$ will become the negative of what it was before. Hence it is evident, that if we write $-x$ for x in any equation, it amounts to reversing the positive direction of the axis of x. And in the same way, if we write $-y$ for y, we reverse the positive direction of the axis of y.

109. Cor. In the same manner, by writing $-\theta$ for θ, we reverse the positive direction of the vectorial angle.

Prop. XXI.

110. To transform the co-ordinates from one set of oblique axes to another having the same origin.

Let OX, OY, and OX', OY' be the two sets of axes (fig. 46), OM, MP, and OM', $M'P$, or x, y, and x', y' the co-ordinates of any point P referred to them; draw $M'Q$ parallel to OX and $M'R$ parallel to OY. Then we have

$$\begin{aligned} OM &= OR + M'Q \\ &= OM' . \frac{\sin(x'y)}{\sin(xy)} + M'P . \frac{\sin(y'y)}{\sin(xy)} . \end{aligned}$$

[We mean by $\sin(xy)$ the sine of the angle which the axis of x makes with the axis of y, and by $\sin(x'y)$ the sine of the angle which the axis of x' makes with the axis of y; and so with the rest.]

Similarly, $MP = RM' + QP$

$$= OM' \frac{\sin(x'x)}{\sin(xy)} + M'P \frac{\sin(y'x)}{\sin(xy)}.$$

Hence, if we assume $\angle(xy) = \omega$, $\angle(x'x) = a$, $\angle(y'x) = \beta$, we have

$$x = x' \frac{\sin(\omega - a)}{\sin \omega} + y' \frac{\sin(\omega - \beta)}{\sin \omega},$$

$$y = x' \frac{\sin a}{\sin \omega} + y' \frac{\sin \beta}{\sin \omega};$$

which are the formulæ necessary to effect the required transformation.

111. Cor. If we wish at the same time to transfer the origin to a point (hk), we have only to add h and k to these expressions for x and y.

Prop. XXII.

112. In the case of polar co-ordinates, to transfer the pole to any point (hk), and turn the prime radius through any angle a.

It is easy to see that we effect this transformation by means of the formulæ

$$r \cos \theta = h + r' \cos(\theta' + a),$$

$$r \sin \theta = k + r' \sin(\theta' + a),$$

which will enable us to change an equation between r and θ into one between r' and θ'; it being evident that r' and θ' are the polar co-ordinates referred to the pole (hk), and to a prime radius making an angle a with the original prime radius.

The use of these various ways of transforming co-ordinates will appear as we go on; we shall often have occasion to refer to Props. XVII., XVIII., XIX., and XX.

Various Problems respecting Lines referred to Oblique Co-ordinates.

113. If O (fig. 47) be the middle point of the side YY' of any triangle YXY', and if we draw $Y'P$, YP' through any point M of OX; to shew that PP' is parallel to YY', and that OX is divided harmonically at the points M and N, (*i.e.* the reciprocals of OM, ON, and OX are in arithmetical progression).

Take OX and OY as co-ordinate axes, and let $OY = OY' = b$, $OM = a$, $OX = a'$; then the equations of $Y'P$ and YX are respectively

$$\frac{x}{a} + \frac{y}{-b} = 1,$$

$$\frac{x}{a'} + \frac{y}{b} = 1.$$

If we suppose x and y to have the same values in these equations, (xy) is the point P; therefore, adding the two equations, which give

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) = 2 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

the value of x we determine is the abscissa of P.

Now, in exactly the same way we find the abscissa of P' by adding the equations of YP' and $Y'X$, viz.

$$\frac{x}{a} + \frac{y}{b} = 1,$$

$$\frac{x}{a'} + \frac{y}{-b} = 1;$$

which give

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) = 2 \ldots\ldots\ldots\ldots\ldots (2);$$

(1) and (2) shew that the points P and P' have the same abscissa, and therefore that PP' is parallel to YY'. Q. E. D.

Again, (1) gives

$$\frac{1}{a} + \frac{1}{a'} = \frac{2}{x},$$

which shews that $\frac{1}{x}$ is an arithmetic mean between $\frac{1}{a}$ and $\frac{1}{a'}$, and therefore that a, x, a' are in harmonical progression. Q. E. D.

114. If OX, OY (fig. 48) be any two right lines intersecting a set of diverging right lines PB, PB', PB'', &c., and if OX be divided harmonically by them; then will OY also be divided harmonically by them; *i.e.* if OA, OA', OA'', &c. form a harmonical progression, so also will OB, OB', OB'', &c.

Let $OA = a$, $OA' = a_1$, $OA'' = a_2$, &c., $OB = b$, $OB' = b_1$, &c.; and let (hk) be the point P. Then the equations of AB, $A'B'$, $A''B''$, &c. are

$$\frac{x}{a} + \frac{y}{b} = 1,$$

$$\frac{x}{a_1} + \frac{y}{b_1} = 1,$$

&c.. . . .&c.

and since these all meet in P, we have

$$\frac{h}{a} + \frac{k}{b} = 1,$$

$$\frac{h}{a_1} + \frac{k}{b_1} = 1,$$

&c.. . . .&c.

By subtracting any one of these equations from that which follows it, we find

$$\left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right) h + \left(\frac{1}{b_{n-1}} - \frac{1}{b_n}\right) k = 0 \ldots\ldots\ldots (1).$$

Now, since a, a_1, a_2, &c. are in harmonical progression, $\frac{1}{a_{n-1}} - \frac{1}{a_n}$ is an invariable difference for all values of n; hence, by (1), the same is true of $\frac{1}{b_{n-1}} - \frac{1}{b_n}$, and therefore b, b_1, b_2, &c. are also in harmonical progression. Q. E. D.

Hence, if OX be cut harmonically, every line drawn from O is also cut harmonically by the diverging lines.

115. OX, OY, and BA (fig. 49) are three given lines, and $A'B'$ is any other line cutting them in A', P, B'; if $OA' + OB'$ be always equal to $OA + OB$, then will $B'P : B'A'$ be an invariable ratio.

Take OA and OB as axes, and let $OA = a$, $OB = b$, $OA' = a'$, $OB' = b'$; then the equations of AB and $A'B'$ are

$$\frac{x}{a} + \frac{y}{b} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{x}{a'} + \frac{y}{b'} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Supposing x and y to have the same values in both these equations, which amounts to supposing (xy) to be the point P, we find, eliminating y,

$$x\left(\frac{b}{a} - \frac{b'}{a'}\right) = b - b';$$

but, since $a + b = a' + b'$, we have

$$\frac{b}{a} - \frac{b'}{a'} = \frac{ba' - b'a}{aa'} = \frac{b(a + b - b') - b'a}{aa'}$$

$$= \frac{(b - b')(a + b)}{aa'}.$$

Hence
$$\frac{x}{a'} = \frac{a}{a + b}.$$

Now $x : a' :: B'P : B'A'$; hence $B'P : B'A'$ is an invariable ratio.

116. If ABC be a triangle having its angles A and B always upon two fixed lines OX, OY, and if the three lines which form the sides of the triangle always pass through three fixed points which are situated in the same right line; then the angle C will always be found upon a fixed line passing through O.

Let the equations of AB, AC, and BC, referred to OX and OY as axes, be

$$\alpha x + \beta y = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\alpha x + \beta' y = 1 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\alpha' x + \beta y = 1 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Observing that, since (1) and (2) meet in OX, they must give the same value of x when $y = 0$; and for a similar reason (1) and (3) must give the same value of y when $x = 0$.

Let (hk), $(h'k')$ and $(h''k'')$ be the three points through which (1), (2) and (3) always pass respectively, and let the equation of the line in which these points are situated be

$$mx + ny = 1 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Then, since hk is a point on (1) and (4), we have, subtracting (4) from (1),
$$(\alpha - m)\,h + (\beta - n)\,k = 0\,;$$
and similarly, since $(h'k')$ and $(h''k'')$ are points on (2) and (4) and on (3) and (4) respectively, we find
$$(\alpha - m)\,h' + (\beta' - n)\,k' = 0,$$
$$(\alpha' - m)\,h'' + (\beta - n)\,k'' = 0.$$

These three equations shew that, whatever be the positions of the lines (1), (2), (3), the quantities $(\alpha - m)$, $(\beta - n)$, $(\alpha' - m)$ and $(\beta' - n)$ are to each other in invariable ratios depending upon the quantities hk, $h'k'$, $h''k''$. Hence it follows that $(\alpha - m) - (\alpha' - m) : (\beta - n) - (\beta' - n)$ is an invariable ratio, $A : B$ suppose; and therefore we have
$$\frac{\alpha - \alpha'}{\beta - \beta'} = \frac{A}{B} \quad\quad (5).$$

Now, supposing (xy) to be the point of intersection of the lines (2) and (3), we have, subtracting (3) from (2),
$$(\alpha - \alpha')\,x - (\beta - \beta')\,y = 0,$$
which, by (5), becomes
$$Ax - By = 0 \quad\quad (6).$$

Hence the point of intersection of (2) and (3), *i.e.* the point C, is always situated on a fixed line passing through the origin, whose equation is (6).

117. Hence the truth of the following theorem is immediately evident, viz.

If ABC, $A'B'C'$ be two triangles such that the point of intersection of AB and $A'B'$, of AC and $A'C'$, and of BC and $B'C'$, lie in the same right line; then the three lines AA', BB', CC' will meet in the same point.

CHAPTER V.

GENERAL EQUATION OF THE CIRCLE. DIAMETER, TANGENT, AND NORMAL OF A CIRCLE. VARIOUS PROBLEMS RESPECTING CIRCLES AND RIGHT LINES.

Prop. XXIII.

118. To find the equation of a circle whose centre and radius are given.

Let (hk) be the centre, a the radius, and (xy) any point of the circumference; then the distance of (xy) from (hk) must be a, and therefore, by Art. 38, we have

$$(x-h)^2+(y-k)^2=a^2 \quad \ldots\ldots\ldots\ldots (1),$$

which, being a relation between the co-ordinates of any point of the circle, is the equation required.

119. Cor. 1. If $h=0$, $k=0$, *i.e.* if the centre be chosen as origin, (1) becomes

$$x^2+y^2=a^2.$$

120. Cor. 2. If $h=a$, $k=0$, *i.e.* if a diameter be chosen as axis of x, and one of its extremities as origin, (1) becomes

$$y^2=2ax-x^2.$$

121. Cor. 3. The equation (1) may represent any circle whatever, and therefore it is the general form of the equation of a circle. If we expand each term, it becomes

$$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0;$$

and hence it appears that the general equation of a circle is of the form

$$x^2+y^2+Ax+By+C=0 \quad \ldots\ldots\ldots\ldots (2),$$

A, B, C being any arbitrary constants. The equation

$$Ax^2+Ay^2+Bx+Cy+D=0 \quad \ldots\ldots\ldots (3)$$

may be immediately reduced to the form (2) by dividing it by

the coefficient of x^2 and y^2. (3) is the most general form in which the equation of a circle can occur, the co-ordinates being rectangular.

PROP. XXIV.

122. *Conversely.* To determine the locus represented by the equation
$$x^2 + y^2 + Ax + By + C = 0.$$

Completing the two squares by adding $\frac{A^2}{4}$ and $\frac{B^2}{4}$ to both sides of this equation, we have

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2}{4} + \frac{B^2}{4} - C.$$

Now the first member of this equation is the square of the distances of the point (xy) from the point $\left(-\frac{A}{2}, \quad -\frac{B}{2}\right)$: hence the locus is a circle whose centre and radius are

$$\left(-\frac{A}{2}, \quad -\frac{B}{2}\right) \text{ and } \sqrt{\left(\frac{A^2}{4} + \frac{B^2}{4} - C\right)}.$$

123. For example, let the equation be
$$x^2 + y^2 - 2cx + 6cy + 9c^2 = 0.$$
Completing the squares, we have
$$(x - c)^2 + (y + 3c)^2 = c^2,$$
which shews that the centre of the circle is $(c, -3c)$, and the radius c.

Again, let the equation be
$$x^2 + y^2 - 2cx - 2cy + 2c^2.$$
Completing the squares, we have
$$(x - c)^2 + (y - c)^2 = 0.$$
Here the centre is (cc), but the radius is zero; therefore the locus is the single point (cc).

Lastly, let the equation be
$$x^2 + y^2 - 2cx + 4cy + 8c^2 = 0.$$
Completing the squares, we have
$$(x - c)^2 + (y + 2c)^2 = -3c^2.$$
Here the radius is impossible, and therefore the equation does not represent any locus.

PROP. XXV.

124. To find the equation of a circle referred to oblique co-ordinates.

Expressing the distance of the point (xy) from the point (hk), by Art. 87, we have

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega=a^2,$$

which is the equation required.

125. COR. If the centre be the origin, the equation is

$$x^2+y^2+2xy\cos\omega=a^2.$$

PROP. XXVI.

126. To find the equation of a circle referred to polar co-ordinates.

Let r and θ be the co-ordinates of any point of the circumference, and b and β those of the centre: then, by Art. 97, we have

$$r^2+b^2-2rb\cos(\theta-\beta)=a^2,$$

which is the equation required.

127. COR. 1. If $b=0$, *i.e.* if the centre be origin, the equation becomes

$$r=a.$$

COR. 2. If $b=a$ and $\beta=0$, *i.e.* if a diameter be the prime radius and one extremity of it the pole, the equation becomes

$$r=2b\cos\theta.$$

Diameters, Tangents, and Normals of a Circle.

PROP. XXVII.

128. To determine the length (r) of a right line drawn, at an angle θ to the axis of x, from a point (hk) to the circumference of a circle $(x^2+y^2=a^2)$, (see Art. 68).

Proceeding exactly as in the Article just referred to, the equation of the circle, when $h+r\cos\theta$ and $k+r\sin\theta$ are put for x and y, becomes

$$(h+r\cos\theta)^2+(k+r\sin\theta)^2=a^2,$$

or

$$r^2+2(h\cos\theta+k\sin\theta)r+h^2+k^2-a^2=0\;\ldots\ldots(1),$$

which equation determines the required distance (r).

Cor. 1. Since this is a quadratic equation, it follows that the right line must, in general, meet the circle in two points, whose distances from (hk) are the two roots of (1).

129. Cor. 2. Let P (fig. 50) be the point (hk), and Q, Q' the two points where the right line meets the circle; then PQ and PQ' are the two roots of (1), and therefore

$$PQ + PQ' = -2(h\cos\theta + k\sin\theta) \;\ldots\ldots\; (2),$$

$$PQ \,.\, PQ' = h^2 + k^2 - a^2 \;\ldots\ldots\ldots\ldots\ldots\; (3).$$

130. Cor. 3. From (3) it follows that, if through any point P a secant be drawn meeting the circle in Q and Q', the rectangle under the segments PQ and PQ' is invariable in whatever direction the secant be drawn. (*Euclid*, Book III. Props. 35 and 36.)

131. Cor. 4. If R be the middle point of the chord QQ', we have $2PR = PQ + PQ'$, and therefore, by (2),

$$PR = -(h\cos\theta + k\sin\theta) \;\ldots\ldots\ldots\; (4).$$

132. Cor. 5. Hence, if P and R coincide, in which case $PR = 0$, we have

$$h\cos\theta + k\sin\theta = 0 \;\ldots\ldots\ldots\ldots\; (5),$$

which is therefore the condition necessary in order that (hk) may be the middle point of the chord which makes an angle θ with the axis of x.

This appears also from the fact that, if (5) hold, the equation (1) gives

$$r = \pm\sqrt{(a^2 - h^2 - k^2)},$$

which shews that one value of r is the negative of the other, and this cannot be unless P be midway between Q and Q'.

Prop. XXVIII.

133. If a chord of a circle be supposed to move parallel to itself,* to determine the locus of its middle point.

Let θ be the invariable angle which the chord makes with the axis of x, and let (xy) be its middle point; then, by the 5th Cor. of the preceding proposition, we have

$$x\cos\theta + y\sin\theta = 0,$$

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* *i. e.* always making the same angle with the axis of x.

which, being a relation between the co-ordinates of the middle point of the chord in any of its positions, is the equation of the required locus. Hence the locus is a right line, evidently passing through the origin, and making an angle whose tangent is $-\cot\theta$ $\left(i.e.\text{ an angle }\theta+\frac{\pi}{2}\right)$ with the axis of x.

134. Cor. The locus of the middle points of a system of parallel chords is called the *diameter* of those chords; hence, by what we have just proved, the diameter of any system of parallel chords in a circle, is a right line passing through the centre, and perpendicular to the chords.

Prop. XXIX.

135. To find the angle which the line touching a circle at a given point makes with the axis of x.

If a right line cutting a circle be supposed to move parallel to itself, until the two points of section coincide and become one point, it is then said to *touch* the circle at that point, or to be the *tangent* of the circle at that point.

If we suppose Q and Q' (in Art. 129) to coincide, we have $PQ = PQ'$, and therefore, by equation (2) in that article, we have

$$PQ = -(h\cos\theta + k\sin\theta),$$

which, since the secant is now become a tangent at the point Q, is the distance of any point (P) on a tangent from the point of contact (Q), θ being the angle which the tangent makes with the axis of x. Now suppose P to coincide with Q, in which case $PQ = 0$, and (hk) becomes the point of contact; then we have

$$h\cos\theta + k\sin\theta = 0, \quad \text{or} \quad \tan\theta = -\frac{h}{k},$$

which is therefore the tangent of the angle which the line touching the circle at the point (hk) makes with the axis of x.

136. Cor. 1. Hence the equation of the tangent at the point (hk) is, by Art. 61,

$$h(x-h) + k(y-k) = 0,$$

which, since $h^2 + k^2 = a^2$, (hk) being a point of the circle, becomes

$$hx + ky = a^2 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

137. COR. 2. To determine the condition necessary in order that the line $(ax + \beta y = \gamma)$ may touch the circle $(x^2 + y^2 = a^2)$.

Let (hk) be the point of contact, then the equation $ax + \beta y = \gamma$ must be identical with (1),* and therefore we have

$$\frac{h}{a^2} = \frac{a}{\gamma} \quad \frac{k}{a^2} = \frac{\beta}{\gamma},$$

and hence
$$\frac{a^2 + \beta^2}{\gamma^2} = \frac{h^2 + k^2}{a^2} = \frac{1}{a^2},$$

or
$$(a^2 + \beta^2)\, a^2 = \gamma^2,$$

which is the condition required.

COR. 3. Hence, if the line $x \sin\theta - y \cos\theta = c$ be a tangent to the circle, we have

$$c^2 = a^2 (\cos^2\theta + \sin^2\theta) = a^2, \quad \text{or } c = \pm a;$$

and therefore the equation becomes

$$x \sin\theta - y \cos\theta = \pm a,$$

which is the equation of the tangent in terms only of the angle it makes with the axis of x. This is a useful form of the equation in problems where there is no occasion to bring the point of contact into consideration. The double sign indicates that two different tangents may be drawn making the same angle with the axis of x, which is manifestly the case.

PROP. XXX.

138. To find the equations of the normal at any point (hk) of a circle.

The right line drawn through the point of contact at right angles to the tangent is called the *normal*.

The equation of the line passing through the point (hk) at right angles to the tangent $(hx + ky = a^2)$ is, by Art. 63, and Art. 60,

$$-k(x - h) + h(y - k) = 0,$$

or
$$kx - hy = 0,$$

which is therefore the equation required. It shews that the

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* In order that two equations $Ax + By = C$ and $A'x + B'y = C'$, may be identical, it is *not* necessary that A, B, C shall be respectively *equal* to A', B', C, but only that they shall be respectively *proportional* to each other.

normal always passes through the centre, and therefore it follows that the tangent at any point is at right angles to the radius drawn to that point. (*Euclid*, Book III. Prop. 16.)

PROP. XXXI.

139. Two tangents may be drawn to a circle from any point (hk) without it; and the equation of the right line joining the two points of contact is $$hx + ky = a^2 \ldots\ldots\ldots\ldots\ldots (1).$$

For, let a right line drawn through (hk) touch the circle at the point $(h'k')$; then the equation of this line, by Art. 136, is $h'x + k'y = a^2$, and this equation must be satisfied when h and k are put for x and y, since (hk) is a point on the line; therefore we have, observing that $(h'k')$ is a point on the circle,

$$h'h + k'k = a^2 \ldots\ldots\ldots\ldots\ldots (2),$$
$$h'^2 + k'^2 = a^2 \ldots\ldots\ldots\ldots\ldots (3).$$

If in (3) we substitute for k' its value given by (2), we evidently find a quadratic which gives two values of h'; and then (2) gives two corresponding values of k'. Hence there are two points of contact, and therefore two tangents may be drawn from (hk) to the circle.

Now let $(h''k'')$ be the other point of contact thus determined from (2) and (3); then we have, by (2),

$$h''h + k''k = a^2 \ldots\ldots\ldots\ldots\ldots (4).$$

Hence it appears that $(h'k')$ and $(h''k'')$ are points on the right line represented by the equation

$$hx + ky = a^2 \ldots\ldots\ldots\ldots\ldots (5);$$

for (2) and (4) shew that (5) is satisfied when either h' and k' or h'' and k'' are substituted for x and y.

Hence it follows that (5) is the equation of the right line joining the points of contact. Q. E. D.

PROP. XXXII.

140. Supposing a chord of a circle to be drawn through a fixed point, and to turn round it; to find the locus of the intersection of the two tangents drawn at its extremities.

Let (xy) be the point of intersection of the two tangents, and (hk) the fixed point through which the line joining the points of

contact always passes; then, by the result (equation 5) of the last proposition (observing that (hk) in equation (5) is the point of intersection of two tangents, and (xy) any point on the line joining their points of contact), we have, putting (hk) for (xy), and (xy) for (hk), $$hx + ky = a^2.$$

This, being a relation between the co-ordinates of the point of intersection of the two tangents drawn at the extremities of any chord passing through (hk), is the equation of the locus required. The locus is therefore a right line, evidently perpendicular to the line $\frac{x}{h} - \frac{y}{k} = 0$, *i. e.* to the line joining the centre and the point (hk).*

141. It is remarkable that the equation $(hx + ky = a^2)$ represents three very different lines, namely, the tangent at (hk), the line joining the points of contact of the two tangents drawn through (hk), and the locus of the intersection of the two tangents drawn at the extremities of any chord drawn through (hk). We may observe that in the first case (hk) must be a point on the circle, in the second case outside it, and in the third inside it; and therefore the three cases are perfectly distinct.

Various problems respecting circles and right lines.

142. To find the locus of the middle point of a chord in a circle which always passes through a given point and turns round it.

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* This result may be proved geometrically, thus:

Let C (fig. 52) be the centre of the circle, B the fixed point through which the chords pass, $T'BT$ any one of the chords, and P the intersection of the tangents at its extremities; produce CB, draw PA perpendicular to it, and let PC meet TT' in Q. Then, since CTP and CQT are right angles, we have $CQ \,.\, CP = CT^2$; and, since Q and A are right angles, we have $CQ \,.\, CP = CB \,.\, CA$; hence

$$CA = \frac{CT^2}{CB} = \text{a constant quantity.}$$

Therefore P is always found on a right line drawn through a fixed point A at right angles to CA.

Let (xy) be the middle point of the chord, and θ the angle it makes with the axis of x; then, by Art. 132, we have

$$x \cos \theta + y \sin \theta = 0 \quad \ldots\ldots\ldots\ldots (1).$$

Also, if (hk) be the given point through which the chord always passes, (xy) must be a point on the line drawn through (hk) and making an angle θ with the axis of x; therefore

$$\frac{y-k}{x-h} = \tan \theta = -\frac{x}{y}, \text{ by (1)},$$

$$\therefore \quad y^2 - ky + x^2 - hx = 0,$$

which, by Art. 122, represents a circle whose centre is $\left(\frac{h}{2}\ \frac{k}{2}\right)$ and $(\text{radius})^2\ \frac{h^2+k^2}{4}$. Hence, if C be the centre of the original circle, and P the point (hk), we have only to describe a circle on CP as diameter, and it will be the locus required.

143. Having given the equations of two circles, to find the equation of the right line passing through their points of intersection.

Let the equations of the two circles be

$$x^2 + y^2 + Ax + By = C' \quad \ldots\ldots\ldots\ldots (1),$$
$$x^2 + y^2 + A'x + B'y = C' \quad \ldots\ldots\ldots\ldots (2),$$

then, subtracting these equations, we find

$$(A - A')x + (B - B')y = C - C' \quad \ldots\ldots\ldots (3),$$

which is the equation of the line required; for we have obtained (3) from (1) and (2) on the supposition that x and y have the same values in (1) and (2), *i. e.* that (xy) is either of the points of intersection of the two circles; therefore (3) is satisfied by the co-ordinates of either point, and therefore the right line represented by (3) is the line drawn through the two points of intersection of (1) and (2). See Art. 139.

144. To shew that if a circle of variable radius be described, touching a given line at a given point, and cutting a given circle, the line drawn through the points of intersection always passes through a given point.

Take the given line as axis of x, and the point where the

circle touches it as origin; then, if r be the variable radius of that circle, its equation is

$$x^2 + y^2 - 2ry = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also, let the equation of the fixed circle be

$$x^2 + y^2 - 2hx - 2ky = c \quad \ldots\ldots\ldots\ldots \quad (2);$$

then, by the previous problem, the equation of the line joining the points of intersection of (1) and (2) is

$$hx + (k - r)y + \frac{c}{2} = 0.$$

Now this line manifestly cuts the axis of x at a distance $-\dfrac{2h}{c}$ from the origin, whatever be the value of r; therefore the line joining the points of intersection always passes through the given point $\left(-\dfrac{c}{2h}, 0\right)$.

145. To find the locus of a point (P) the sum of the squares of whose distances from a given set of points (A, A', A'', &c.) is given.

Let (xy) be the point P, and (hk), $(h'k')$ &c. the points A, A', A'', &c.; then we have

$$(x-h)^2 + (y-k)^2 + (x-h')^2 + (y-k')^2 + (x-h'')^2 + (y-k'')^2 + \&c.$$
$$= \text{a given quantity, } C \text{ suppose.}$$

Therefore, if n be the number of the given points, and if we put

$$h + h' + h'' + \&c. = H, \qquad k + k' + k'' + \&c. = K,$$
$$C - (h^2 + h'^2 + h''^2 \ldots) - (k^2 + k'^2 + k''^2 \ldots) = L,$$

we find $$nx^2 + ny^2 - 2Hx - 2Ky = L;$$

hence the locus required is a circle.

146. To find the locus of the vertex of a triangle whose base and ratio of sides are given.

Take the given base as axis of x and its middle point as origin, let $2a$ be its length, and $m : n$ the ratio of the sides; then, if (xy) be the vertex, the lengths of the sides are $\sqrt{\{(x-a)^2 + y^2\}}$ and $\sqrt{\{(x+a)^2 + y^2\}}$; and we therefore have

$$\frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} = \frac{m^2}{n^2},$$

or $$(x^2 + y^2)(n^2 - m^2) - 2a(n^2 + m^2)x + a^2(n^2 - m^2).$$

This is the equation of a circle whose centre is on the axis of x at a distance $a \cdot \frac{n^2 + m^2}{n^2 - m^2}$, from the origin, and whose radius is $a \cdot \frac{2nm}{n^2 - m^2}$.

147. A right line is drawn in any direction from a given point to the circumference of a given circle; to find the locus of the point which divides it in a given ratio.

Take the given point as origin, and let the equation of the given circle be $(x - h)^2 + (y - k)^2 = a^2 \ldots\ldots\ldots\ldots (1)$;

then if $(x'y')$ be the co-ordinates of the point which divides any line drawn from the origin to this circle in a given ratio, it is evident that the co-ordinates of a point of the circle will be mx', my', m being a given ratio. Therefore these values put for x and y must satisfy (1); and consequently we have

$$(mx' - h)^2 + (my' - k)^2 = a^2,$$

or, neglecting the dashes,

$$\left(x - \frac{h}{m}\right)^2 + \left(y - \frac{k}{m}\right)^2 = \frac{a^2}{m^2},$$

which represents a circle whose centre is $\left(\frac{h}{m}\ \frac{k}{m}\right)$ and radius $\frac{a}{m}$. This problem may be easily solved by using the polar equation of the circle.

148. To find the locus of the vertex of a triangle whose base and vertical angle are given.

149. To find the locus of the centre of the inscribed circle, under the same circumstances.

150. To find the locus of the intersection of the perpendiculars let fall from the extremities of the base upon the opposite sides, under the same circumstances.

151. To find the locus of the intersection of the lines drawn from the extremities of the base bisecting the opposite sides, under the same circumstances.

If θ and θ' be the angles which the sides make with the base (taken as axis of x and its middle point as origin), it is easy to

shew that the equations of the two bisectors are

$$(2\cot\theta' - \cot\theta)y = x + a\ldots(1), \quad (2\cot\theta - \cot\theta')y = x - a\ldots(2);$$

and if α be the given vertical angle, we have $\theta - \theta' = \alpha$, and therefore

$$\tan\theta - \tan\theta' = \tan\alpha(1 + \tan\theta\tan\theta')\ldots\ldots(3).$$

Now, if we suppose x and y to have the same values in (1) and (2), and therefore (xy) to be the point of intersection of the two bisectors; we find, from (1) and (2),

$$\tan\theta' = \frac{3y}{3x - a}, \qquad \tan\theta = \frac{3y}{3x + a},$$

and hence by (3)

$$3y(-2a) = \tan\alpha(9x^2 - a^2 + 9y^2),$$

$$\text{or} \qquad x^2 + y^2 + \frac{2}{3}a\cot\alpha \,.\, y = \frac{a^2}{9},$$

which represents a circle.

152. Given the base and the sum of the sides of a triangle, to find the locus of the intersection of the bisector of the vertical angle with the perpendicular upon it from either extremity of the base.

153. To determine the same when the difference of the sides instead of their sum is given.

154. A line of given length is placed with its extremities on the two co-ordinate axes, to determine the locus of its middle point.

155. An isosceles triangle is placed with the extremities of its base on the two co-ordinate axes, to determine the locus of its vertex.

156. If two lines, whose lengths are in a given ratio, be drawn from a given point and contain a given angle, and if the extremity of one be always on a given right line, or on a given circle, to find the locus of the extremity of the other.

Take the given point as pole, let r, r' be the two lines, and θ, θ' the angles they make with any line through the given point taken as prime radius; let α be the constant angle contained by r and r', and $m:1$ the given ratio of $r:r'$. Then if the extremity of r' be on a given right line, whose equation is

$$r' = \frac{c\sin\beta}{\sin(\beta - \theta')},$$

we have (since $r = mr'$ and $\theta' = \theta + a$),

$$r = \frac{mc \sin \beta}{\sin (\beta - a - \theta)};$$

which shews that the locus is a right line making an angle $\beta - a$ with the prime radius, and meeting it at a distance $\dfrac{mc \sin \beta}{\sin (\beta - a)}$ from the pole. See Art. 158.

If the extremity of r' be on a circle whose equation is

$$a^2 = r'^2 + c^2 - 2r'c \cos (\theta' - \beta),$$

we find $\quad m^2a^2 = r^2 + m^2c^2 - 2mc \,.\, r \cos \{\theta - (\beta - a)\},$

which shews that the locus is a circle whose radius is ma, and the co-ordinates of whose centre are mc and $\beta - a$.

157. To determine the loci represented by the following equations, by transforming them to rectangular equations.

$$nr \sin (\theta - a) + mr \cos (\theta + a) + c = 0 \;\ldots\ldots (1),$$

$$cr \sin (\theta - a) - cr \cos (\theta + a) + mr^2 + b^2 = 0 \;\ldots\ldots (2).$$

If (1) be transformed to rectangular co-ordinates by putting $r \cos \theta = x$, $r \sin \theta = y$, it becomes

$$(n \cos a - m \sin a)\, y - (n \sin a - m \cos a)\, x + c = 0,$$

which represents a right line easily determined in position.

In the same way (2) becomes

$$m (x^2 + y^2) + c (\sin a + \cos a)(y - x) + b^2 = 0,$$

which represents a circle easily determined in position and magnitude.

158. It is worth remarking that the position of a right line represented by a polar equation may be easily determined by finding the value of θ when $r = \infty$ and the value of r when $\theta = 0$; the former being the angle the line makes with the prime radius, and the latter the portion it cuts off from the prime radius.

Thus, if the equation be $r \sin (\theta + \beta) = c \cos \beta$, we find $r = \infty$ when $\theta = -\beta$, and $r = c \cot \beta$ when $\theta = 0$; therefore the line makes an angle $-\beta$ with the prime radius, and cuts off from it the portion $c \cot \beta$.

159. To find the general equation of a right line by means of the method of transformation of co-ordinates.

The equation of a right line parallel to the axis of y is $x = c$; and, if we turn the axes through an angle $-\theta$, this becomes

$$y \cos\theta - x \sin\theta = c,$$

which is therefore the equation of a right line making any angle θ with the axis of x and at a perpendicular distance c from the origin.

160. To determine what the equation, $5(x^2 + y^2) - 6xy = 4a^2$, becomes when the axes are turned through an angle 45°.

161. Transform the equation, $r(1 + e\cos\theta) = l$, from polar into rectangular co-ordinates; and thence into polar co-ordinates again, the pole being removed to a distance $\dfrac{el}{e^2 - 1}$ from its original position.

162. To describe a circle passing through a given point and touching two given lines.

Take the two given lines as axes of co-ordinates, let ω be the angle they make with each other, and (hk) the given point; let the equation of the circle required be

$$x^2 + y^2 + 2xy\cos\omega - 2Ax - 2By + C = 0 \quad \ldots. \quad (1).$$

Then, since the circle touches the axis of x, the values of x given by this equation when $y = 0$ must be equal; therefore, putting $y = 0$, the first member of the equation, which becomes

$$x^2 - 2Ax + C = 0,$$

must be a perfect square; which gives

$$A^2 = C;$$

and in the same way we may shew that

$$B^2 = C.$$

Hence (1) becomes

$$x^2 + y^2 + 2xy\cos\omega - 2A(x \pm y) + A^2 = 0 \quad \ldots. \quad (2).$$

Now, (hk) being a point on this circle, we have

$$h^2 + k^2 + 2hk\cos\omega - 2A(h \pm k) + A^2 = 0 \quad \ldots. \quad (3),$$

(3) determines A, and then (2) is completely given. The value of A got from (3) is

$$A = h \pm k \pm \sqrt{\{(h \pm k)^2 - (h^2 + k^2 + 2hk\cos\omega)\}},$$

which is impossible unless we reject the lower of the two signs in $(h \pm k)$; we have therefore

$$A = h + k \pm 2 \sin \frac{\omega}{2} \sqrt{(hk)}.$$

Hence two circles may fulfil the conditions of the problem.

163. To describe a circle touching a given right line and passing through two given points.

164. To describe a circle with its centre on a given right line and touching two given circles.

CHAPTER VI.

OF THE LOCI CALLED CONIC SECTIONS, THEIR FORMS, AND EQUATIONS.

In the previous chapters we have applied the method of co-ordinates to the investigation of various properties of right lines and circles; we now proceed to employ the same method in the case of certain loci, commonly called *Conic Sections*, which rank next in importance and interest to the right line and circle.

Prop. XXXIII.

165. To determine the locus of a point (P) (fig. 53) whose distance (SP) from a given point (S) is always proportional to its perpendicular distance (PQ) from a given line (EK).

Draw the right line XSE perpendicular to EK, and through any point (O) of SE draw OY perpendicular to SE; take OX, OY as co-ordinate axes, let OM $(= x)$, MP $(= y)$ be the co-ordinates of P, and put $OS = m$, $OE = n$. Then we have

$$PQ = ME = x + n, \quad PS^2 = (x - m)^2 + y^2;$$

and hence, since $PS \propto PQ = e \,.\, PQ$, suppose, e being some given ratio, we have

$$(x - m)^2 + y^2 = e^2 (x + n)^2,$$

or
$$(1 - e^2)\, x^2 - 2\,(m + e^2 n)\, x + y^2 = e^2 n^2 - m^2,$$

which is the equation of the required locus.

Since O is arbitrary we may assume it to be the point which makes $m = en$ (*i.e.* the point which divides ES in the given ratio $QP : SP$); and then the equation becomes

$$(1 - e^2)\, x^2 - 2\,(1 + e)\, mx + y^2 = 0 \quad \ldots\ldots \quad (1).$$

We shall now determine the form of the locus by means of this equation, and in doing this we shall consider three cases, namely, where e is less than unity, where e is equal to unity, and where e is greater than unity.

PROP. XXXIV.

166. To trace the form of the locus when e is less than unity.

Dividing (1) by $1 - e^2$, it becomes

$$x^2 - 2\frac{m}{1-e}x + \frac{y^2}{1-e^2} = 0;$$

assume, for brevity, $\frac{m}{1-e} = a$, add a^2 to both sides, and we have

$$(x-a)^2 + \frac{y^2}{1-e^2} = a^2;$$

transfer the origin to the point (a, o), which is done by writing $x + a$ for x (see Art. 105), divide by a^2, and for brevity put $a^2(1-e^2) = b^2$; then the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

This is the equation of the locus referred to a new origin O' (fig. 54), OO' being equal to a, the value of a being $\frac{m}{1-e}$. We have thus transformed the equation in order to simplify it, and make it easier to trace its form, which we now proceed to do.

From (2), we find $y = \pm b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$;

hence, if we take $O'M$ (fig 54) as any value of x, there will be two corresponding values of y, one the negative of the other; if, therefore, MP be one of these values, and if we produce it downwards till $MP' = MP$, then MP' is the other value. It appears therefore that any value of x ($O'M$) gives two points of the curve (P and P') equidistant from the axis of x, one above and the other below it. Hence it follows that the curve is divided by the axis of x into two parts, which are exactly similar, but reversed in position; or, in other words, the curve is perfectly symmetrical with respect to the axis of x.

By solving the equation (2) for x instead of for y we may shew in the same manner that the curve is perfectly symmetrical with respect to the axis of y also.

If we suppose M to move from O' in the positive direction, *i.e.* if we suppose x to increase continually from the value zero, the above expression for y shews that MP will continually

diminish, having its greatest value (b) when M is at O', and becoming zero when $O'M = a$; beyond which MP is impossible, and therefore the point P has no existence.

Hence, if we assume $O'A = a\,(= O'O)$ and $O'B = O'B' = b$, it is evident that the form of the curve is that represented in the figure.

Prop. XXXV.

167. To trace the form of the locus when e is equal to unity.

The equation (1) in this case becomes

$$y^2 = 4mx \;\ldots\ldots$$

which gives $$y = \pm\, 2\,\sqrt{(mx)}.$$

Hence, as in the previous case, we may shew that the curve is perfectly symmetrical with respect to the axis of x.

If we suppose M to move from O in the positive direction, this expression for y shews that MP continually increases, being zero when M is at O. When M is any where on the negative side of O, x is negative, and therefore y is impossible; which shews that P has no existence then.

Hence the form of the curve is evidently that represented in fig. 55.

In this case we could not have transformed (1) into the form (2), for since $e = 1$, $a = \infty$, and therefore the new origin (O') is at an infinite distance from the old (O).

Prop. XXXVI.

168. To trace the form of the locus when e is greater than unity.

In this case $1 - e$ and $1 - e^2$ are negative quantities; we shall therefore (in order to avoid negative and impossible quantities) put the equation (1) in the form

$$x^2 + 2\,\frac{m}{e-1}\,x - \frac{y^2}{e^2-1} = 0\,;$$

and we shall assume $\dfrac{m}{e-1} = a$, and $a^2\,(e^2 - 1) = b^2$. Hence, adding a^2 to both sides and dividing by a^2, the equation becomes

$$\frac{x + a)^2}{a^2} - \frac{y^2}{b^2} = 1\,;$$

and removing the origin to the point $(-a, o)$, which is done by writing $x - a$ for x, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

This is the equation of the locus referred to a new origin O' (fig. 55) where $OO' = -a$, the value of a being $\frac{m}{e-1}$ and that of b^2 being $a^2(e^2 - 1)$.

We may shew, exactly as in the first case, that the curve is perfectly symmetrical with respect to the axes of x and y. Also, since from (3) we have

$$y = \pm b \sqrt{\left(\frac{x^2}{a^2} - 1\right)},$$

it is evident that, supposing M to move from O' in the positive direction, MP is impossible while M is between O' and O, is zero when M is at O, and continually increases when M is beyond O.

Hence the form of the curve is that represented in fig. 55.

169. Thus it appears that the equation (1) represents three curves of very different form, according as x is less than, equal to, or greater than unity. In the first case the curve is called an *ellipse*, in the second a *parabola*, and in the third a *hyperbola*. The origin of these names was as follows.

170. Let $OHH'M$ be a rectangle (fig. 56) constructed upon the abscissa AM, and having its *erect side* OH equal to the coefficient of x in equation (1), namely, $2(1+e)m$; which quantity has on this account acquired the name of the *latus rectum*. Then $2(1+e)mx$ is equal to this rectangle, and therefore, by (1), we have $\quad MP^2 - OHH'M = (e^2 - 1)x^2.$

Hence it is evident that, according as e is less than, equal to, or greater than unity, the square of the ordinate falls short of, is equal to, or exceeds the rectangle under the abscissa and latus rectum. It was in reference to this defect, equality, or excess, that the names ellipse, parabola, hyperbola, were given to the three curves we have been considering.

171. The point S (Art. 165) is called the *focus*, and the line EK the *directrix*; S was formerly termed *punctum comparationis*.

The ratio e is called the *eccentricity*, the reason of which name will appear as we go on. Hence the eccentricity is less than unity in the ellipse, greater than unity in the hyperbola, and equal to unity in the parabola.

If we take $O'S'$ equal to $O'S$ (figs. 54 and 55) and $O'E' = O'E$, it is evident, from the perfect symmetry of the curve with respect to the axis of y, that the point S', and the right line $(E'K')$ drawn through E' at right angles to $O'E'$ have exactly the same properties, with reference to the curve, as the point S and the line EK; and therefore the distance of any point of the curve from S' is to its perpendicular distance from $E'K'$ in the ratio $e : 1$. There are therefore *two* foci and *two* corresponding directrices in the ellipse and hyperbola.

172. The point O' is called the *centre*, on account of the symmetry of the curve with respect to the two axes which intersect in O'. We shall shew that O' bisects every line drawn through it in both directions to the curve. It is usual to denote the centre by the letter C, the point O by the letter A, and the other point where the curve crosses the axis of x by A'. The line AA' (which $= 2a$) is called the *axis*, and the points A and A' the *vertices* of the curve. C is the middle point of AA'.

173. In the ellipse OO' is drawn in the positive direction, and in the hyperbola in the negative, as we have shewn above. The parabola has no centre, since OO' is infinite; and it meets the axis of x only at O; therefore its axis is infinite, and it has but one vertex.

174. In the ellipse the points where the curve crosses the axis of y are usually denoted by the letters B and B'; the line BB' (which $= 2b$) is bisected by the centre. Since $\frac{b^2}{a^2} = 1 - e^2$ (see Art. 166) and e is < 1, b is $> a$; on this account AA' (or $2a$) is called the *major* axis of the ellipse, and BB' (or $2b$) the *minor* axis.

175. In the hyperbola the curve does not cross the axis of y; on this account AA' is called the possible axis of the hyperbola, and the axis of y is called the impossible axis. Since

$\frac{b^2}{a^2} = e^2 - 1$ in the hyperbola (Art. 168), and e may be of any magnitude greater than unity, it is evident that b may be either greater or less than a in any proportion.

176. Figs. 57, 58, 59 represent the three curves marked with their proper letters; to which notation we shall always adhere. It is important to bear in mind the following relations which are assumed in Arts. 166 and 168; viz.

$$AS = m \quad AC = \frac{m}{1-e} = a \text{ in the ellipse,}$$
$$= -a \text{ in the hyperbola;}$$
$$\frac{b^2}{a^2} = 1 - e^2 \text{ in the ellipse,}$$
$$= e^2 - 1 \text{ in the hyperbola;}$$

and, if we denote the latus rectum by $2l$, we have

$$l = (1 + e)\, m.$$

From these formulæ we find the following for the ellipse,

$$SC = ae, \quad EC = \frac{a}{e}, \quad l = a(1 - e^2) = \frac{b^2}{a};$$

for, since $AS = m = a(1 - e)$, we have

$$SC = AC - AS = a - a(1 - e) = ae,$$

also
$$EC = EA + AC = \frac{m}{e} + a = \frac{a}{e},$$

and
$$l = m(1 + e) = a(1 - e^2) = \frac{b^2}{a}.$$

The corresponding formulæ in the hyperbola are

$$SC = -ae, \quad EC = -\frac{a}{e}, \quad l = a(e^2 - 1) = \frac{b^2}{a}.$$

Prop. XXXVII.

177. To determine the polar equation of the three curves referred to S as pole and SX prime radius.

Using the notation and figure of Art. 165, and putting $SP = r$, $PSX = \theta$, we have

$$QP = EM = n + m + r\cos\theta;$$

and hence, since $SP = eQP$, and $en = m$, we find

$$r = m(1 + e) + er\cos\theta;$$

and therefore $$r = \frac{m(1+e)}{1-e\cos\theta} = \frac{l}{1-e\cos\theta} \quad\ldots\ldots\ldots (1),$$
which is the polar equation required.

If we reverse the positive direction of the angle θ, *i.e.* if we put $ASP = \theta$ instead of $A'SP$, the polar equation becomes

$$r = \frac{l}{1+e\cos\theta}.$$

178. Cor. When $\theta = \frac{\pi}{2}$, r becomes the ordinate drawn through S; but, by (1), $r = l$ when $\theta = \frac{\pi}{2}$. Hence the latus rectum ($2l$) is the double ordinate drawn through S.

179. The three curves we have been considering are of considerable importance; we often meet with them in various parts of mathematics, especially in Physical Astronomy. It is therefore necessary to be acquainted with their properties, many of which are very remarkable and well worthy of attention for their own sake, independently of any use that may be made of them. We shall devote several chapters of the present treatise to the investigation of these properties.

We often meet with equations of the form

$$Ax^2 + By^2 = C, \quad Ay^2 = Bx, \quad Ax^2 = By,$$

where A, B, C are any constants, positive or negative; and it is important to be able to make out exactly the loci which these equations represent, and how far they depend in kind and position upon the values of the constants. On this account the following propositions are added.

Prop. XXXVIII.

180. To determine the locus represented by the equation

$$Ax^2 + By^2 = C.$$

Dividing this equation by C, it becomes

$$\frac{A}{C}x^2 + \frac{B}{C}y^2 = 1,$$

which, according as $\frac{A}{C}$ and $\frac{B}{C}$ happen to be positive or negative,

may be put in one of the following forms, viz.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots (1), \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots (3),$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots\ldots (2), \qquad -\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots\ldots (4).$$

(1) and (2) we have already considered; but we must observe respecting (1) that b may be greater than a, which is not consistent with Art. 174. We have however, in this case, only to interchange the co-ordinate axes by writing x for y and y for x, and to assume $\frac{A}{C} = \frac{1}{b^2}$ and $\frac{B}{C} = \frac{1}{a^2}$, instead of $\frac{A}{C} = \frac{1}{a^2}$ and $\frac{B}{C} = \frac{1}{b^2}$. In this manner the equation will be reduced to the proper form. (1) therefore represents an ellipse whether b be less than a or greater than a, only in the latter case the axis of y is the axis on which the two foci are situated; fig. 60 therefore represents the locus in the latter case.

(3) may be reduced to the same form as (2) by interchanging the axes of co-ordinates; therefore it represents a hyperbola having its foci on the axis of y. (See fig. 61.)

(4) does not represent any locus, because all real values of x and y make the first side negative, whereas the second side is positive.

181. If C be zero we cannot divide the given equation by it, but we may determine the locus thus. The equation becomes $Ax^2 + By^2 = 0$, which, according as A and B have the same or contrary signs, may be put in either of the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \ldots\ldots (1), \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \ldots\ldots (2).$$

The first member of (1) is the sum of two essentially positive quantities, which therefore cannot together make zero, unless each be separately zero. Hence (1) gives $x = 0$, $y = 0$, and these are the only values of x and y which satisfy (1). It therefore represents the isolated point (oo), *i.e.* the origin.

(2) may be put in the form

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0\,;$$

and therefore (2) is satisfied by all values of x and y which make either of these factors zero, and by no other values. Now if (xy) be any point on the right line $\left(\frac{x}{a} - \frac{y}{b} = 0\right)$ the first factor is zero, and if (xy) be any point on $\left(\frac{x}{a} + \frac{y}{b} = 0\right)$ the second factor is zero. Hence it is evident that the equation (1) represents the two right lines

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0.$$

182. If in Art. 165 we suppose m to be zero the equation becomes

$$(1 - e^2)\,x^2 + y^2 = 0,$$

which is of the same form as (1) or (2) (in the present Art.) according as e is $<$ or $>$ 1. We may therefore regard the point and the two right lines represented by (1) and (2) as an ellipse and a hyperbola in which the points S and A coincide.

Prop. XXXIX.

183. To determine the locus represented by the equations

$$Ay^2 = Bx, \qquad Ax^2 = By.$$

These equations, according as A and B have the same or different signs, may be put in one of the following forms, viz.

$$y^2 = 4mx \;\ldots\ldots\; (1), \qquad x^2 = 4my \;\ldots\ldots\; (3),$$

$$y^2 = -\,4mx \;\ldots\ldots\; (2), \qquad x^2 = -\,4my \;\ldots\ldots\; (4).$$

(1) we have already considered. (2) may be reduced to the same form as (1) by writing $-x$ for x, *i.e.* by reversing the positive direction of the axis of x (see Art. 108); and therefore (2) represents a parabola situated as in fig. 62. (3) may be reduced to the same form as (1) by interchanging the co-ordinate axes; and (4) may be reduced to the form (3) by reversing the positive direction of the axis of y. Hence the loci (3) and (4) are parabolæ situated as in figs. 63 and 64.

184. If $m = 0$ (1) and (2) represent the axis of x, and (3) and (4) the axis of y. If $e = 1$ and $m = 0$ in the equation obtained in Art. 165, it becomes $y^2 = 0$; therefore a parabola becomes a right line when the points S and A coincide.

The object of the following chapter is to shew that the loci we have just been considering are represented by the general equation of the second degree between x and y, and that they are also identical with the curves produced by the section of a common cone by a plane in different positions. We shall afterwards investigate the various remarkable properties of the ellipse, parabola, and hyperbola separately. This being the case, the following chapter may be passed over on a first perusal of this subject.

CHAPTER VII.

OF THE GENERAL EQUATION OF THE SECOND DEGREE BETWEEN x AND y, AND THE LOCI IT REPRESENTS. REDUCTION OF THE GENERAL EQUATION TO ITS SIMPLEST FORM. SECTIONS OF A RIGHT CONE BY A PLANE ARE LOCI OF THE SECOND ORDER.

185. We have seen that the general equation of the first degree between x and y represents a right line; we shall now consider the general equation of the second degree, and shew how to determine the locus it represents in any given case.

The general equation of the second degree between x and y contains the terms x^2, xy, y^2, x, and y, each multiplied by some constant, together with a term independent of x and y; it may therefore be put in the form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

some of the constants being multiplied by 2 (since we may assume arbitrary constants in any form we please), in order to simplify certain formulæ we shall obtain hereafter.

We shall first suppose that the term $2Bxy$ is wanting in this equation, and on this supposition determine the nature of the locus. We shall then shew that the term $2Bxy$, if it occur in the equation, may be made to disappear out of it, by simply turning the axes of co-ordinates through a certain angle; and thus we shall reduce the general case to the particular case previously considered.

Prop. XL.

186. To determine the nature of the locus represented by the general equation of the second degree, supposing that the term $2Bxy$ is wanting.

In this case the equation is

$$Ax^2 + Cy^2 + 2Dx + 2Ey + F = 0 \ldots\ldots\ldots (1).$$

1st, Suppose that neither A nor C are zero; then we may assume $D = Ah$, $E = Ck$; and, substituting these values, (1) becomes

$$A(x^2 + 2hx) + C(y^2 + 2ky) = -F;$$

or, adding $Ah^2 + Ck^2$ to each side, and putting $Ah^2 + Ck^2 - F = G$ for brevity, we have

$$A(x+h)^2 + C(y+k)^2 = G;$$

and if we transfer the origin to the point $(-h, -k)$, which is done by writing $x - h$ and $y - k$ for x and y, this equation becomes

$$Ax^2 + Cy^2 = G \ldots\ldots\ldots\ldots (2).$$

2ndly, Suppose that $A = 0$; then, assuming $E = Ck$, the equation (1) may be put in the form

$$C(y+k)^2 = Ck^2 - F - 2Dx,$$

and, assuming $Ck^2 - F = -2Dh$ (supposing D not equal to zero), we have

$$C(y+k)^2 = -2D(x+h),$$

which, if we transfer the origin to the point $(-h, -k)$, becomes

$$Cy^2 = -2Dx \ldots\ldots\ldots\ldots (3).$$

If however $D = 0$, then (1) becomes

$$Cy^2 + 2Ey + F = 0 \ldots\ldots\ldots\ldots (4).$$

3rdly, Suppose that $C = 0$; then the equation (1) may be reduced, as in the previous case, to either of the forms

$$Ax^2 = -2Ey \ldots\ldots\ldots\ldots (5),$$

$$Ax^2 + 2Dx + F = 0 \ldots\ldots\ldots\ldots (6).$$

The equations (2), (3), (5) we have already fully considered in Arts. 180–184. The equation (4) may be put in the form

$$(y-p)(y-q) = 0,$$

which, if p and q be real, represents (see Art. 181) the two right lines $(y = p)$ and $(y = q)$ which are parallel to the axis of x; and if p and q be impossible, it does not represent any locus. In the same manner (6) represents either two parallel lines or no locus.

Hence it appears that the general equation of the second degree wanting the term $2Bxy$ represents an ellipse, a parabola,

a hyperbola, two right lines, one right line, or an isolated point; except in certain cases where it does not represent any locus.

The following proposition will complete the discussion of the general equation.

PROP. XLI.

187. To shew that the term $2Bxy$ may be made to disappear from the general equation of the second degree by turning the axes of co-ordinates through a certain angle.

If, in the general equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \ldots. (1),$$

we write $x \cos\theta - y \sin\theta$ in place of x, and $x \sin\theta + y \cos\theta$ in place of y, we turn the axes through the angle θ, leaving the origin unaltered (see Art. 104). When these substitutions for x and y are made, (1) may evidently be expanded and arranged so as to assume the form

$$A'x^2 + 2B'xy + C'y^2 + 2D'x + 2E'y + F = 0 \ldots. (2),$$

A', B', C', D', &c. being quantities formed from A, B, C, D, E, and θ, in arranging the equation in the form (2).

$2B'xy$ is the only term of (2) we have occasion to know for our present purpose; it can only arise from the three first terms of (1), which, after the substitutions for x and y, become

$$A(x\cos\theta - y\sin\theta)^2 + 2B(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + C(x\sin\theta + y\cos\theta)^2,$$

or $A'x^2 + 2\{-A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + C\sin\theta\cos\theta\}xy + C'y^2$.

Hence $$2B' = 2B\cos 2\theta - (A - C)\sin 2\theta.$$

Now, from this expression, it appears that we may make $B' = 0$, by putting

$$\tan 2\theta = \frac{2B}{A - C} \ldots\ldots\ldots\ldots (3),$$

and this we may always do, since the tangent of an angle may be of any magnitude, finite or infinite, positive or negative.

Hence it appears that, when θ is determined from (3), the term $2B'xy$ does not occur in (2). Consequently, by turning the axes of co-ordinates through a certain angle, determined by (3), we may always make the term $2Bxy$ disappear from the general equation of the second degree. Q. E. D.

188. COR. Hence, and by the previous proposition, the most general equation of the second degree represents an ellipse, a parabola, a hyperbola, two right lines, one right line, an isolated point, or no locus.

189. By means of this and the previous proposition we may determine the form and position of the locus represented by any given equation of the second degree. We must first get rid of the term $2Bxy$, if it occur in the equation, by determining the proper value of θ from (3), and substituting it in A', C', D', and E'; and then we have only to reduce the resulting equation as in Prop. XL., and so find the locus required. But this process is rather tedious, and we may determine the locus much more readily by means of the following proposition.

PROP. XLII.

190. If the equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \;\ldots\; (1)$$

be reduced to the form

$$A'x^2 + C'y^2 + 2D'x + 2E'y + F = 0 \ldots\ldots\ldots (2),$$

by turning the axes of co-ordinates through an angle θ; to determine A', C', D', E', and θ.

Since (1) is reduced to (2) by turning the axes through the angle θ, it follows that (2) is reduced to (1) by turning the axes to which (2) is referred through the angle $-\theta$. Therefore, if we put $x\cos(-\theta) - y\sin(-\theta)$ for x, and $x\sin(-\theta) + y\cos(-\theta)$ for y in (2), it must become identical with (1). Making these substitutions in (2), it becomes

$$A'(x\cos\theta + y\sin\theta)^2 + C'(-x\sin\theta + y\cos\theta)^2 + 2D'(x\cos\theta + y\sin\theta) \\ + 2E'(-x\sin\theta + y\cos\theta) + F = 0.$$

Hence, equating the coefficients of x^2, xy, y^2, x and y in this equation to those in (1), we have

$$A'\cos^2\theta + C'\sin^2\theta = A \ldots\ldots\ldots\ldots (3),$$

$$(A' - C')\sin\theta\cos\theta = B \ldots\ldots\ldots\ldots (4),$$

$$A'\sin^2\theta + C'\cos^2\theta = C \ldots\ldots\ldots\ldots (5),$$

$$D'\cos\theta - E'\sin\theta = D \ldots\ldots\ldots\ldots (6),$$

$$D'\sin\theta + E'\cos\theta = E \ldots\ldots\ldots\ldots (7).$$

The first three of these equations determine A', C', and θ, and then the two last give D' and E'; and thus we obtain the five required quantities.

191. A few examples will shew the use of this proposition.

Ex. 1. To determine the locus represented by the equation

$$5x^2 + 2xy + 5y^2 - 12c\sqrt{2}y - 12c\sqrt{2}x = 0.$$

In this case the above equations become

$$A' \cos^2\theta + C' \sin^2\theta = 5 \ldots\ldots\ldots\ldots(1),$$
$$(A' - C') \sin\theta \cos\theta = 1 \ldots\ldots\ldots\ldots(2),$$
$$A' \sin^2\theta + C' \cos^2\theta = 5 \ldots\ldots\ldots\ldots(3),$$
$$D' \cos\theta - E' \sin\theta = -6c\sqrt{2} \ldots\ldots\ldots(4),$$
$$D' \sin\theta + E' \cos\theta = -6c\sqrt{2} \ldots\ldots\ldots(5).$$

(1) - (3) gives $\quad (A' - C') \cos 2\theta = 0,$

which, since $A' - C'$ is not zero by (2), gives $\theta = 45°$; and therefore (1) and (2) become

$$A' + C' = 10, \quad A' - C' = 2;$$

and therefore $\quad A' = 6, \quad C' = 4.$

Also (4) and (5) become

$$D' - E' = -12c,$$
$$D' + E' = -12c,$$

and therefore $\quad D' = -12c, \quad E' = 0.$

Hence, by turning the axes through an angle 45°, the given equation becomes

$$6x^2 + 4y^2 - 24cy = 0,$$

or $\quad 3x^2 + 2(y - 6c)^2 = 72c^2,$

which, if we transfer the origin to the point $(0, 6c)$, becomes

$$\frac{x^2}{24c^2} + \frac{y^2}{36c^2} = 1;$$

and this is the equation of an ellipse whose axes are $2c\sqrt{6}$ and $6c$.

Hence, if we take $X'OX = 45°$ (fig. 65), draw OC $(= 6c)$ at right angles to OX', and CA parallel to OX'; then the ellipse described on CA and CB as semiaxes is that represented by the given equation, where $CA = 2c\sqrt{6}$, $CB = 6c$.

192. Ex. 2. Let the given equation be

$$x^2 + 2xy - y^2 - 2cx + 2cy - 4c^2 = 0.$$

In this case we have

$$A' \cos^2 \theta + C' \sin^2 \theta = 1 \text{ ....(1)},$$
$$(A' - C') \sin \theta \cos \theta = 1 \text{ ....(2)},$$
$$A' \sin^2 \theta + C' \cos^2 \theta = -1 \text{ ....(3)},$$
$$D' \cos \theta - E' \sin \theta = -c \text{ ....(4)},$$
$$D' \sin \theta + E' \cos \theta = c \text{ ....(5)}.$$

(1) − (3) gives $\qquad (A' - C') \cos 2\theta = 2 \text{ ....(6)}.$

2 (2) ÷ (6) gives $\quad \tan 2\theta = 1, \quad \text{and} \therefore 2\theta = 45°.$

Hence (6) and (1) + (3) become

$$A' - C' = 2\sqrt{2}, \qquad A' + C' = 0,$$

and therefore $\qquad A' = \sqrt{2}, \qquad C' = -\sqrt{2}.$

Hence, if we turn the axes through an angle $22\frac{1}{2}°$, and put $D' = \sqrt{2}h$, $E' = \sqrt{2}k$, the given equation becomes

$$\sqrt{2} \,.\, x^2 - \sqrt{2}\, y^2 + 2\sqrt{2}hx + 2\sqrt{2}ky - 4c^2 = 0,$$

or $\qquad (x + h)^2 - (y - k)^2 = 2\sqrt{2}c^2 + h^2 - k^2.$

Now, by (4) and (5),

$$\sqrt{2}h = -c(\cos \theta - \sin \theta), \qquad \sqrt{2}k = c(\cos \theta + \sin \theta),$$

and therefore

$$2(h^2 - k^2) = c^2(-4 \sin \theta \cos \theta) = -c^2\sqrt{2}.$$

Hence, transferring the origin to the point (hk), we have

$$x^2 - y^2 = \frac{3}{\sqrt{2}}\, c^2,$$

which represents a hyperbola. h and k we have found above in terms of θ. We shall presently give a simpler method of determining (hk).

193. Ex. 3. Let the given equation be

$$x^2 - 2xy + y^2 - 4\sqrt{2}cx = 0.$$

Here we find, just as in Ex. 1,

$$\theta = 45°, \quad A' + C' = 2, \quad A' - C' = -2,$$

and therefore $\qquad A' = 0, \quad C' = 2.$

Also $\qquad D' - E' = -4c, \quad D' + E' = 0,$

and therefore $\qquad D' = -2c, \quad E' = 2c.$

Hence, by turning the axes through an angle 45°, the given equation becomes

$$2y^2 - 4cx + 4cy = 0,$$

or

$$(y + c)^2 = 2c\left(x - \frac{c}{2}\right),$$

which, if we transfer the origin to the point $\left(\frac{c}{2}, -c\right)$, becomes

$$y^2 = 2cx.$$

Hence the given equation represents a parabola.

194. Ex. 4. Let the given equation be

$$x^2 - 2xy + y^2 - 2c^2 = 0.$$

Here $\theta = 45°$, $A' = 0$, $C' = 2$, $D' = 0$, $E' = 0$; therefore, by turning the axes through an angle 45°, the equation becomes

$$y^2 - c^2 = 0, \quad \text{or} \quad (y - c)(y + c) = 0,$$

which represents two parallel lines. Indeed this may be seen immediately, for the given equation may be put in the form

$$(x - y - c\sqrt{2})(x - y + c\sqrt{2}) = 0,$$

which represents two parallel right lines making an angle 45° with the axis of x.

Prop. XLIII.

195. To shew that the general equation of the second degree represents, in general, an ellipse, parabola, or hyperbola, according as $AC - B^2$ is positive, zero, or negative.

It is evident, from Art. 186, that the equation (2) in Art. 190, represents, in general, an ellipse, if A' and C' have the same sign; a parabola if either A' or C' be zero, and a hyperbola if A' and C' have opposite signs; *i.e.* the locus is an ellipse, parabola, or hyperbola, according as $A'C'$ is positive, zero, or negative. Now, referring to the equations for determining θ, A', C', &c. in Art. 190, we may thus determine $A'C'$.

(3) × (5) gives

$$(A'^2 + C'^2)\sin^2\theta\cos^2\theta + A'C'(\cos^4\theta + \sin^4\theta) = AC \quad \dots (8),$$

and (8) − (4)² gives

$$A'C'(2\sin^2\theta\cos^2\theta + \cos^4\theta + \sin^4\theta) = AC - B^2,$$

or

$$A'C' = AC - B^2.$$

Hence it follows that $A'C'$ is positive, zero, or negative, according as $AC - B^2$ is so; and therefore the truth of the proposition is manifest.

PROP. XLIV.

196. To determine the general values of θ, A', C', D', E' from the equations obtained in Art. 190.

(3) − (5) gives $\quad (A' - C') \cos 2\theta = A - C \ldots\ldots\ldots\ldots (9),$

and $2(4) \div (9)$ gives

$$\tan 2\theta = \frac{2B}{A - C},$$

which is the same value of $\tan 2\theta$ as that found in Art. 187.

Again we find, as in the preceding article,

$$A'C' = AC - B^2.$$

Also (3) + (5) gives $\quad A' + C' = A + C.$

Hence A' and C' are the two values of z given by the quadratic equation $\quad z^2 - (A + C)z + AC - B^2.$

Lastly, the values of D' and E' obtained from the equations (6) and (7), are

$$D' = D \cos \theta + E \sin \theta, \quad E' = E \cos \theta - D \sin \theta.$$

Thus the values of θ, A', C', D', and E', are completely determined.

PROP. XLIV. (*bis.*)

192 (*bis*). To cause the terms $2Dx$, $2Ey$ to disappear from the general equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \ldots\ldots (1).$$

Transfer the origin to any point (hk), and the equation becomes

$$A(x+h)^2 + 2B(x+h)(y+k) + C(y+k)^2 + 2D(x+h) + 2E(y+k) + F = 0,$$

or

$$Ax^2 + 2Bxy + Cy^2 + 2(Ah+Bk+D)x + 2(Ck+Bh+E)y + F' = 0 \ldots (2),$$

where $\quad F' = Ah^2 + 2Bhk + Ck^2 + 2Dh + 2Ek + F.$

Now if we assume, as we may do,

$$Ah + Bk + D = 0 \ldots (3), \quad Ck + Bh + E = 0 \ldots (4),$$

(which two equations determine h and k), the equation (2) becomes $\quad Ax^2 + 2Bxy + Cy^2 + F' = 0 \ldots\ldots\ldots\ldots (5).$

Thus we have made the terms $2Dx$ and $2Ey$ to disappear by transferring the origin to the point (hk), h and k being given by the equations (3) and (4).

193 (*bis*). Cor. 1. $(3)\,h + (4)\,k$ gives

$$Ah^2 + 2Bhk + Ck^2 + Dh + Ek = 0\,;$$

therefore the value of F' becomes $Dh + Ek + F$. Hence it appears that, if we transfer the origin to the point (hk) determined by the equations (3) and (4), the equation (1) becomes

$$Ax^2 + 2Bxy + Cy^2 + Dh + Ek + F = 0 \ldots\ldots (6).$$

194 (*bis*). Cor. 2. The values of h and k, given by (3) and (4), are

$$h = -\frac{DC - EB}{AC - B^2}, \quad k = -\frac{EA - DB}{AC - B^2}.$$

Hence, by Art. 195, h and k are infinite in the case of the parabola.

195 (*bis*). Cor. 3. If in the equation (6) we put $x = r\cos\theta$, $y = r\sin\theta$, we evidently find $r = \pm\sqrt{P}$ (where P is a quantity depending upon θ, the precise value of which we have no occasion to know). Hence every value of θ gives two values of r, one the negative of the other; and therefore every chord drawn through the origin is bisected at the origin. It appears therefore that the curve is perfectly symmetrical with respect to the origin, and consequently the origin must be its *centre*. Hence the point (hk), determined by the equations (3) and (4), is the centre of the locus represented by (1).

196 (*bis*). In those cases where $AC - B^2$ is not zero, the present proposition will enable us to simplify the application of the method given in Prop. XLII.: for, by first reducing the general equation to the form $Ax^2 + 2Bxy + Cy^2 + F' = 0$, and then applying the method alluded to, we shall have no occasion to employ the equations (6) and (7), (Art. 190), since D' and E' will evidently be each zero.

For example, let us take the equation in Art. 192, namely

$$x^2 + 2xy - y^2 - 2cx + 2cy - 4c^2 = 0.$$

If we transfer this to its centre, it becomes, by equations (3), (4), (6) in the present Prop.,

$$x^2 + 2xy - y^2 - ch + ck - 4c^2 = 0 \ldots\ldots\ldots (7),$$

where $\qquad h + k - c = 0, \quad -k + h + c = 0;$

and therefore $h = 0$, $k = c$; which being substituted in (7), it becomes

$$x^2 + 2xy - y^2 - 3c^2 = 0.$$

Now, proceeding exactly as in Art. 192, observing that D and E and therefore D' and E' are zero, we find

$$A = \sqrt{2}, \quad C = -\sqrt{2},$$

and therefore $$x^2 - y^2 = \frac{3}{\sqrt{2}} c^2,$$

which is the same equation as that obtained in Art. 192. We have here determined the co-ordinates of the centre to be 0 and c, whereas in Art. 192 we only obtained them in terms of θ (which $= 22\frac{1}{2}^\circ$).

In cases where $AC - B^2 = 0$, h and k are infinite (in general), and we must therefore proceed altogether as in Prop. XLII.

Of the Sections of a Right Cone by a Plane.

The ellipse, parabola, and hyperbola, are aften called *lines of the second order*, because they are represented, as we have seen, by the general equation of the second degree between x and y. But they are more commonly known by the name of *conic sections*, because they are the curves produced by the section of a cone by a plane, as we now proceed to shew.

PROP. XLV.

197. To shew that the section of the surface of a right cone made by any plane is always a line of the second order.

A *right cone* is the solid generated by the revolution of a right-angled triangle round one of its sides. By the surface of this cone we mean the surface generated by the revolution of the hypotenuse produced indefinitely both ways. The side round which the triangle turns is called the axis of the cone. If any plane be supposed to cut the cone, the curve in which the conical surface meets that plane is termed the section of the conical surface made by it. When the cutting plane is perpendicular to the axis of the cone, it is evident that the section is

a circle, its centre being the point where the axis meets the plane.

Let ABC (fig. 66) represent the cone, and $OPQP'$ the section made by the cutting plane; let $RPSP'$ be the circular section made by any plane perpendicular to the axis, and let $ARMQ$ be the plane containing the axis of the cone, and perpendicular to PMP', the line of section of the two former planes; then PM is perpendicular to OM and RM, and RS is a diameter of the circle $RPSP'$.

Take OQ as axis of x, and let $OM(=x)$ and $MP(=y)$ be the co-ordinates of P, which may be any point of the section $OPQP'$; then, since $PM^2 = RM.MS$ by a property of the circle, we have

$$y^2 = RM.MS \quad\ldots\ldots\ldots\ldots\ldots(1).$$

Now, drawing ON and TQ (fig. 67) parallel to RS, we have

$$\frac{RM}{TQ} = \frac{OM}{OQ}, \quad \frac{MS}{ON} = \frac{MQ}{OQ} \quad\ldots\ldots\ldots (2).$$

Hence, if we put $ON = c$, $ROQ = \theta$, $OAQ = 2a$, $OQ = 2a$, and therefore

$$TQ = OQ\,\frac{\sin TOQ}{\sin ATQ} = 2a\,\frac{\sin\theta}{\cos a},$$

we have, by (2), since $OM = x$, and $MQ = 2a - x$,

$$RM.MS = \frac{c}{2a}\,\frac{\sin\theta}{\cos a}\,(2ax - x^2).$$

Hence (1) becomes

$$y^2 = \frac{c}{2a}\,\frac{\sin\theta}{\cos a}\,(2ax - x^2) \quad\ldots\ldots\ldots\ldots (3),$$

which is an equation of the second degree between x and y; and therefore the section of a right cone by a plane is always a line of the second order. Q. E. D.

198. Cor. 1. Since $\dfrac{c}{2a} = \dfrac{\sin(\theta - 2a)}{\cos a}$, (3) may be put in the form

$$y^2 = \frac{\sin\theta}{\cos a}\left\{cx - \frac{\sin(\theta - 2a)}{\cos a}\,x^2\right\};$$

or

$$Ay^2 + Bx^2 - Cx = 0,$$

when $A = \cos^2 a$, $B = \sin\theta\sin(\theta - 2a)$, $C = c\sin\theta\cos a$.

Now if the cutting-plane coincide with a line (OT) drawn parallel to AQ, $\theta = 2\alpha$; if it fall on the right of OT, θ is $> 2\alpha$; and if it fall on the left, θ is $< 2\alpha$; and in these cases B is zero, positive, or negative, respectively. Hence, since A is essentially positive, the curve is an ellipse, parabola, or hyperbola, according as the cutting-plane falls on the right of, coincides with, or falls on the left of, the line OT which is drawn parallel to AC.

199. Cor. 2. When the cutting-plane passes through A, it is evident that the section is two right lines, one right line, or a point.

CHAPTER VIII.

OF THE PARABOLA. PROPERTIES OF THE PARABOLA NOT CONNECTED WITH TANGENTS OR DIAMETERS. PROPERTIES CONNECTED WITH TANGENTS. PROPERTIES CONNECTED WITH DIAMETERS. VARIOUS PROBLEMS.

200. We now proceed to investigate in detail the various remarkable properties of the ellipse, parabola, and hyperbola. We shall consider each curve separately, commencing with the parabola, because it is the simplest of the three.

201. The equation of the parabola referred to its vertex A as origin is, by Chap. VI.,

$$y^2 = 4mx \dots\dots\dots\dots\dots\dots (1),$$

m denoting the distance of the focus (S) from the vertex. Since $e = 1$ and $AS = e \,.\, EA$, m is also the distance of the foot of the directrix from the vertex. The general value of the latus rectum is $2m(1+e)$, therefore, putting $e = 1$, the value of the latus rectum in the parabola is $4m$. Hence, in the parabola, the square of the ordinate is a mean proportional between the abscissa and latus rectum.

The polar equation of the parabola referred to S as pole is, putting $e = 1$ in Art. 177,

$$r = \frac{2m}{1 - \cos\theta} \dots\dots\dots\dots\dots\dots (2).$$

From these equations we shall investigate the various remarkable properties of the parabola.

Properties of the Parabola not connected with Tangents or Diameters.

PROP. XLVI.

202. The distance (SP) of any point (xy) of the parabola from the focus (S) is equal to $x + m$.

For, SP being the distance between the points (m, o) and (xy), we have

$$SP^2 = (x - m)^2 + y^2$$
$$= x^2 + 2mx + m^2, \quad \text{since } y^2 = 4mx;$$

therefore $SP = x + m.$* Q. E. D.

203. COR. If we turn the axes round through any angle θ, and transfer the origin to any point (hk); in which case we must, by Art. 106, write $x \cos\theta - y \sin\theta + h$ for x; we find

$$SP = x \cos\theta - y \sin\theta + h + m.$$

Hence, to *whatever* axes of co-ordinates the parabola be referred, the expression for SP is always a *linear function* (Art. 50, Note) of the co-ordinates of P, *i.e.* an expression of the form $Ax + By + C$.

PROP. XLVII.

204. To determine in general the locus of a point (xy), whose distance from a given point (hk) is a linear function of x and y.

Let the expression for the distance of (xy) from (hk) be

$$Ax + By + C = r.$$

Now the formula for the perpendicular distance of the point (xy) from the right line whose equation is

$$Ax_1 + By_1 + C = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

is, by Art. 69, $$\frac{Ax + By + C}{\sqrt{(A^2 + B^2)}} = p;$$

and therefore $r = \sqrt{(A^2 + B^2)} \,.\, p.$

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* This may be also shewn very readily by the polar equation, and geometrically, as follows.

In fig. 68, QP being parallel to AS, we have

$$SP = QP = AM + EA = x + m, \quad \text{since } EA = m.$$

Hence the perpendicular distance of (xy) from the line (1) is always proportional to its distance from the point (hk). Therefore, by Art. 165, the locus of (xy) is a conic section, whose focus is (hk), whose directrix is (1), and whose eccentricity is $\sqrt{(A^2 + B^2)}$.

205. Cor. 1. Hence we may define the focus of a conic section to be that point whose distance from any point (xy) of the curve is a linear function of x and y.

206. Cor. 2. Hence, if the conic section be a parabola referred to its axis as axis of x and its vertex as origin, (1), being the equation of the directrix, gives $B = 0$, $\frac{C}{A} = m$; and, since the eccentricity is unity, we have $A^2 + B^2$ or A^2 equal to 1. Hence

$$r = x + m,$$

which is the result of Art. 202.

Prop. XLVIII.

207. If PSP' be any chord drawn through the focus, then the semi-latus-rectum is a harmonic mean between SP and SP'.

For if $PSX = \theta$ we have, by the polar equation,

$$\frac{1}{SP} = \frac{1 - \cos\theta}{l},$$

and putting $\theta + \pi$ for θ, and therefore SP' for SP, we have

$$\frac{1}{SP'} = \frac{1 + \cos\theta}{l}.$$

Hence

$$\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l},$$

which shews that l is a harmonic mean between SP and SP'.

Prop. XLIX.

208. Two given lines being drawn through the vertex of a parabola, to find the equation of the line joining the other two points in which they intersect the curve.

Let the equations of the two lines be

$$y - ax = 0, \quad y - a'x = 0.$$

Then, as in Art. 181, both the lines together are represented by the equation

$$(y - ax)(y - a'x) = 0,$$

or

$$y^2 + aa'x^2 - (a + a')xy = 0.$$

Where this locus intersects the parabola we have $y^2 = 4mx$; therefore, substituting this value for y^2 and dividing out x, we have

$$4m + aa'x - (a + a')y \;\ldots\ldots\ldots\ldots (1),$$

which equation is satisfied by the co-ordinates of both the points where the two lines again meet the parabola; therefore, reasoning exactly as in Art. 139, (1) is the equation of the right line joining these points.

209. Cor. Putting $y = 0$ in (1), we find $x = -\dfrac{4m}{aa'}$, which, if the two lines be at right angles, and therefore $aa' = -1$, becomes $x = 4m$. Hence, if any two lines be drawn from the vertex of a parabola at right angles to each other, the line joining the two points where they again meet the parabola always crosses the axis at a distance $4m$ from the vertex.

Prop. L.

210. To determine the length (r) of a right line drawn, at an angle θ to the axis of x, from any point (hk) to the parabola.

Proceeding exactly as in Arts. 68, 128, putting $h + r\cos\theta$ and $k + r\sin\theta$ for x and y in the equation of the parabola, we have

$$(k + r\sin\theta)^2 = 4m(h + r\cos\theta),$$

or

$$r^2\sin^2\theta - 2(2m\cos\theta - k\sin\theta)r + k^2 - 4mh = 0 \ldots (1),$$

which determines the required distance.

Since this is a quadratic equation, the right line must in general meet the parabola in two points whose distances from (hk) are the roots of (1).

211. Cor. 1. Using the same notation as in Arts. 129, 130, we have

$$PQ + PQ' = 2\,\frac{2m\cos\theta - k\sin\theta}{\sin^2\theta} \;\ldots\ldots (2),$$

$$PQ \, . \, PQ' = \frac{k^2 - 4mh}{\sin^2\theta} \;\ldots\ldots\ldots\ldots (3).$$

212. Cor. 2. Hence, as in Art. 132, if (hk) be the middle point of QQ', we have

$$2m \cos \theta - k \sin \theta = 0 \ldots\ldots\ldots\ldots (4),$$

which is therefore the condition necessary in order that the point (hk) may be the middle point of a chord which makes an angle θ with the axis of x.

Prop. LI.

213. If QQ', RR' be two chords of a parabola, and P their point of intersection, the ratio $PQ \,.\, PQ' : PR \,.\, PR'$ is not altered by moving each chord parallel to itself, and so shifting the position of P in any manner.

By the previous Article we have

$$PQ \,.\, PQ' = \frac{k^2 - 4mh}{\sin^2 \theta},$$

θ being the angle which QQ' makes with the axis, and (hk) the point P. In the same way, if ϕ be the angle which RR' makes with the axis, we have

$$PR \,.\, PR' = \frac{k^2 - 4mh}{\sin^2 \phi} \ldots\ldots\ldots\ldots (5),$$

therefore $$\frac{PQ \,.\, PQ'}{PR \,.\, PR'} = \frac{\sin^2 \phi}{\sin^2 \theta}.$$

Now if we move each chord parallel to itself in any manner, we do not alter θ or ϕ; hence the truth of the proposition is manifest.

Cor. If either θ or ϕ be equal to 0 or π, we must state this proposition somewhat differently. Suppose that $\theta = 0$, then the equation (1) becomes

$$-4mr + k^2 - 4mh = 0,$$

which shews that the right line meets the parabola in only one point, Q suppose, and that

$$PQ = \frac{k^2 - 4mh}{4m}.$$

Therefore, by (5), we have

$$\frac{PR \,.\, PR'}{PQ} = \frac{4m}{\sin^2 \phi}.$$

Hence, if from any point P of a chord RR' we draw the line PQ to the parabola parallel to the axis, the ratio $PR.PR' : PQ$ is not altered by moving PQ or RR' parallel to itself in any manner.

The remarkable property of two intersecting chords proved in this proposition is true for all the conic sections.

Properties of the Parabola connected with Tangents.

PROP. LII.

214. To determine the equation of the right line which touches a parabola at a given point.

Proceeding exactly as in Arts. 135, 136, if we suppose the right line in Art. 210 to move parallel to itself till Q and Q' coincide, and if we also suppose P to coincide with Q, the equation (2), Art. 211, becomes

$$0 = 2m \cos\theta - k \sin\theta,$$

or

$$\tan\theta = \frac{2m}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

which is the tangent of the angle which the line touching the parabola at the point (hk) makes with the axis.

Hence the equation of this line, since it passes through (hk), is

$$k(y - k) - 2m(x - h) = 0,$$

which, since $k^2 = 4mh$, becomes

$$ky - 2m(x + h) = 0 \ldots\ldots\ldots\ldots\ldots \text{(2)},$$

and this is the equation required.*

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* We may obtain this result somewhat differently in the following manner.

Let (hk), $(h'k')$ be any two points on the parabola; then the equation of the line drawn through these points is

$$y - k = \frac{k' - k}{h' - h}(x - h) \ldots\ldots \text{(4)},$$

Now, subtracting the equations $k^2 = 4mh$, $k'^2 = 4mh'$, we find $k'^2 - k^2 = 4m(h' - h)$, and therefore $\frac{k' - k}{h' - h} = \frac{4m}{k' + k}$; hence the equation becomes

$$y - k = \frac{4m}{k' + k}(x - h);$$

Prop. LIII.

215. If $y = ax + \beta$ be the equation of any tangent of the parabola, to determine what relation must hold between a and β.

Let (hk) be the point of contact, then, by Art. 214, the two equations

$$y = ax + \beta, \quad \text{and } y = \frac{2m}{k}(x + h),$$

must represent the same line, and therefore we have

$$\frac{2m}{k} = a, \quad \frac{2mh}{k} = \beta;$$

therefore $$a\beta = \frac{4m^2h}{k^2} = m, \text{ since } k^2 = 4mh.$$

Hence $a\beta = m$ is the relation required.*

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which equation may be made to differ as little as we please from the equation

$$y - k = \frac{2m}{k}(x - h) \; \ldots\ldots\ (5),$$

by bringing the point $(h'k')$ sufficiently near to the point (hk). Hence the line represented by (4) may be made to approach as near as we please to the line represented by (5), by making $(h'k')$ approach (hk); which evidently cannot be, unless (5) be the tangent at the point (hk). (5) therefore represents the tangent at (hk), and it may be reduced, as in the text, to the form

$$ky = 2m(x + h).$$

It is erroneous to say that the line (4) becomes the line (5) when $(h'k')$ coincides with (hk), for two reasons: 1st, when $(h'k')$ and (hk) coincide, (4) is no longer a definite line, but any line whatever drawn through (hk); 2ndly, we are not at liberty to divide the equation

$$k'^2 - k^2 - 4m(h' - h) = 0,$$

by $h' - h$ when $h' = h$; for there is no rule of Algebra which enables us to divide an equation by zero; and therefore we cannot prove that $\dfrac{k' - k}{h' - h} = \dfrac{4m}{k' + k}$ when $(h'k')$ and (hk) coincide.

* We may prove this result somewhat differently, as follows.

If we eliminate y between the equations $y = ax + \beta$ and $y^2 = 4mx$, we find

$$(ax + \beta)^2 = 4mx, \quad \text{or} \quad a^2x^2 + 2(a\beta - 2m)x + \beta^2 = 0.$$

Now, if the line touch the parabola, this equation, which determines the abscissæ of their points of intersection, ought to give two equal values of x, and therefore its first member ought to be a perfect square; we have therefore

$$a^2\beta^2 = (a\beta - 2m)^2, \quad \text{or} \quad a\beta = m.$$

Hence the equation of a line touching a parabola (in terms simply of the angle it makes with the axis) is

$$y = ax + \frac{m}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

PROP. LIV.

216. To determine the subtangent and subnormal of any point of the parabola.

If T (fig. 69) be the point where the tangent at P meets the axis, the line TM is called the *subtangent;* and if PG be perpendicular to PT, PG is called the *normal,* and MG the *subnormal.*

The point P being (hk) the equation of PT is

$$ky - 2m(x + h) = 0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

and, PG being a line drawn through (hk) at right angles to this line, its equation is

$$2m(y - k) + k(x - h) = 0 \quad \ldots\ldots\ldots\ldots \quad (2).$$

And, if we put $y = 0$, x becomes AT in (1), and AG in (2). Hence

$$AT = -h, \quad \text{or } TA = h, \quad \text{and } AG = 2m + h;$$

and therefore $TM = TA + AM = 2h$, $MG = AG - AM = 2m$.

Hence the subtangent of P is twice the abscissa (AM), and the subnormal twice the distance AS.

217. COR. $TS = TA + AS = h + m$; but, by Art. 202, $SP = m + h$; hence $TS = SP$, therefore the triangle TSP is an isosceles triangle, and the angle TPS is equal to the angle PTS. Hence the tangent at P makes equal angles with SP and the axis; or, if we draw PN parallel to AS, the tangent PT bisects the angle SPT.*

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* We may shew this geometrically, as follows.

Let P' (fig. 70) be any point of the parabola taken very near P, draw PQ, $P'Q'$ parallel to AS, PV perpendicular to QP, and take $SU = SP$. Then, since $SP = PQ$ and $SP' = P'Q'$, we have $VP' = UP'$; also, since $SP = SU$, and U may be taken as near P as we please by making P' approach P, it is evident that the angles at U may be made to differ from right angles as little as we please. Hence in the triangles PUP', PVP', we have PP' common, $P'V = P'U$, and $P'UP$ may be made as nearly equal

218. This may be also shewn as follows.

The equation of TP being $ky - 2m(x+h) = 0$, we have

$$\tan T = \frac{2m}{k};$$

and therefore $\quad \tan 2T = \dfrac{2\dfrac{2m}{k}}{1 - \dfrac{4m^2}{k^2}} = \dfrac{4mk}{4mh - 4m^2} = \dfrac{k}{h-m}.$

Now the equation of SP, a line drawn through (m, o) and (hk), is

$$\frac{y-0}{k-0} = \frac{x-m}{h-m};$$

and therefore $\quad \tan PSX = \dfrac{k}{h-m} = \tan 2T.$

Hence $PSX = 2PTS$, and therefore $PTS = SPT$.

Prop. LV.

219. To find the locus of the intersection of a tangent with the perpendicular let fall upon it from S.

By Art. 215, the equation of the tangent is

$$y = ax + \frac{m}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation of a line at right angles to this through S is

$$y = -\frac{1}{a}(x-m) \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Suppose x and y to have the same values in (1) and (2), and therefore to belong to the point of intersection of the two lines; then (1) – (2) gives

$$\left(a + \frac{1}{a}\right)x = 0,$$

which, since $a + \dfrac{1}{a}$ cannot be zero in general (nor indeed in any case) gives

$$x = 0,$$

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to $P'VP$ as we please, by making P' approach P; therefore $PP'U$ may be made as nearly equal to $PP'V$ as we please, by making P' approach P; which cannot be, unless the tangent at P makes equal angles with PS and PQ.

which, being true for all values of a, shews that the point of intersection of the two lines is always on the axis of y. Hence the axis of y is the locus required.*

220. Cor. 1. This proposition suggests the following method of drawing (geometrically) a tangent, or rather two tangents, to a parabola from any given point U (fig. 72).

Join S and U, on SU as diameter describe a circle cutting AY at R and R'; then the lines drawn from U through R and R' will touch the parabola.

This is evident, since SRU and $SR'U$ are right angles; and therefore, by the proposition just proved, UR and UR' produced must be tangents to the parabola.

221. Cor. 2. If U be on the directrix, the point V where SU meets AY must be the middle point of SU; therefore RR' must be a diameter of the circle, and consequently RUR' must be a right angle. Hence the two tangents drawn from any point of the directrix must be at right angles to each other. This we shall prove analytically in the following proposition.

Prop. LVI.

222. To determine, analytically, the tangents of a parabola which pass through a given point.

The equation of the tangent is

$$y = ax + \frac{m}{a}.$$

Suppose it to pass through a given point (hk), then we have

$$k = ah + \frac{m}{a},$$

or

$$a^2 - \frac{k}{h}a + \frac{m}{h} = 0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

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* This may be shewn geometrically, as follows (fig. 71).

Let PR be the tangent at P meeting QS at R; then, since $PS = PQ$, and $QPR = SPR$, PR must be at right angles to SR; also we have $SR = RQ$, and therefore, since $SA = AE$, AR must be parallel to EQ, *i.e.* AR is the axis of y. Hence the tangent and perpendicular from S upon it intersect in the axis of y.

which equation gives two values of a, and therefore shews that in general two tangents may be drawn through a given point. If a, a' be the two roots of (1), the equations of the two tangents are

$$y = ax + \frac{m}{a} \dots\dots (2), \quad y = a'x + \frac{m}{a'} \dots\dots (3),$$

223. Cor. To determine the locus of the point of intersection of two tangents at right angles to each other.

If (2) and (3) be at right angles, we have

$$aa' + 1 = 0.$$

But, by (1), $aa' = \frac{m}{h}$; hence

$$\frac{m}{h} + 1 = 0, \quad \text{or } h = -m,$$

which shews that the point of intersection of the two tangents is always on the directrix. The directrix is therefore the locus required.

Prop. LVII.

224. The tangent at P, and the right line drawn through S at right angles to SP, meet the directrix at the same point.

Supposing (hk) to be P, the equation of the tangent is

$$ky - 2m(x + h) = 0;$$

and therefore, putting $x = -m$, we find that the ordinate of the point where the tangent meets the directrix is

$$\frac{2m(h - m)}{k}.$$

Now $\frac{k}{h - m}$ is evidently the tangent of the angle which SP makes with the axis, and therefore the equation of the line drawn through S at right angles to SP is

$$y = -\frac{h - m}{k}(x - m);$$

and therefore, putting $x = -m$, we find that the ordinate of the point where this line meets the directrix is

$$\frac{2m(h - m)}{k}.$$

Hence the two lines meet the directrix at the same point.

PROP. LVIII.

225. To find the length of the perpendicular upon the tangent from the focus.

The equation of the tangent being

$$ky = 2m(x + h),$$

the length of the perpendicular upon it from the point (m, o) is, by Art. 69,

$$\frac{2m(m + h)}{\sqrt{(k^2 + 4m^2)}}.$$

Hence, if we represent this length by p, and observe that $k^2 = 4mh$, and $m + h = SP = r$, suppose; we have

$$p^2 = mr,^*$$

which gives the perpendicular upon the tangent in terms of the radius vector of the point of contact.

PROP. LIX.

226. To determine the equation of the line joining the points of contact of the two tangents drawn from a given point to a parabola.

Let (hk) be the given point, and (xy) either of the points of contact; then the equation of the tangent gives

$$ky - 2m(x + h) = 0;$$

and, reasoning just as in Art. 139, we may shew that this is the equation of the line joining the two points of contact. Q. E. F.

COR. 1. Hence, reasoning exactly as in Art. 140, we may shew that the locus of the intersection of the two tangents drawn at the extremities of any chord passing through the point (hk) is

$$ky - 2m(x + h) = 0.$$

227. COR. 2. Hence it appears that the equation,

$$ky - 2m(x + h) = 0,$$

represents three very different lines, namely, 1st, the tangent at the point (hk); 2ndly, the line joining the points of contact of

* This result is easily proved geometrically, thus:

In fig. 71, since $SRP = 90°$, $RAS = 90°$, we have

$$SP : SR = SR : SA, \quad \text{and} \therefore SP \cdot SA = SR^2,$$

which gives $p^2 = mr$.

the two tangents drawn from the point (hk); and 3rdly, the locus of the intersection of the two tangents drawn at the extremities of a chord which always passes through the point (hk).

228. Cor. 3. Hence the locus of the intersection of the two tangents drawn at the extremities of a chord, which always passes through the focus, is the directrix. For, putting $k = 0$, $h = m$ in the equation $ky - 2m(x + h) = 0$, it becomes $x = -m$, which represents the directrix.

229. Cor. 4. Hence, by Art. 223, the two tangents at the extremities of a focal chord are at right angles to each other.

Properties of the Parabola connected with Diameters.

PROP. LX.

230. To determine the diameter of a given system of parallel chords in the parabola, *i.e.* the locus of the middle points of the chords.

Let θ be the angle which any chord makes with the axis, and (xy) its middle point; then, by Art. 212, we have

$$2m \cos \theta - y \sin \theta = 0;$$

and consequently, if we suppose the chord to move parallel to itself, and therefore θ to be invariable, this equation represents the locus of its middle point. Hence the diameter of a system of parallel chords, inclined at an angle θ to the axis, is the right line whose equation is

$$y = 2m \cot \theta,$$

which represents a line parallel to the axis.

231. Cor. If we suppose the chord to move parallel to itself until its extremities coincide, it then becomes a tangent; and hence it follows that the tangent at the extremity of any diameter is parallel to the chords of that diameter. This may also be easily seen from the equations of the diameter and tangent; for if (hk) be the extremity of any diameter, and θ the angle its chords make with the axis, we have, by the equation of the diameter just obtained,

$$\tan \theta = \frac{2m}{k};$$

hence, by the equation of the tangent (Art. 214), θ is also the angle which the tangent at (hk) makes with the axis; therefore the tangent is parallel to the chords.

PROP. LXI.

232. To find the equation of the parabola referred to any diameter as axis of x, and the tangent at its extremity as axis of y.

Let QX', TQY' (fig. 73) be any diameter and the tangent at its extremity, RPR' any chord parallel to QY', and therefore, by the preceding Article, a chord of the diameter QX', and consequently bisected by QX' at P.

Then, by Art. 213, we have

$$\frac{PR \, . \, PR'}{PQ} = \frac{TQ \, . \, TQ}{TA},$$

or $$PR^2 = \frac{TQ^2}{TA} \, . \, PQ, \quad \text{since } PR = PR'.$$

Hence, if we put

$$PQ = x, \quad PR = y, \quad \text{and} \quad \frac{TQ^2}{TA} = 4m', \text{ for brevity,}$$

we find $$y^2 = 4m'x \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1);^*$$

which, being a general relation between the co-ordinates of any

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* This result may be obtained by transformation of co-ordinates, in the following manner.

If in the equation $y^2 = 4mx$ (which is referred to AX, AY) we put for x, $h + x + y \cos\theta$, and for y, $k + y \sin\theta$, we have

$$(k + y \sin\theta)^2 = 4m\,(h + x + y\cos\theta) \quad \ldots\ldots \quad (2);$$

and this substitution amounts to transferring the origin to (hk), and inclining the axis of y so that it may make an angle θ with the axis of x, (this appears easily from fig. 74, where AM and MR are the old x and y, and QP, PR the new; $AN = h$, $NQ = k$, $RPX' = \theta$). (See Art. 110.)

Now, supposing (hk) to be the extremity of any diameter, and θ the angle which its tangent makes with AX, we have $k^2 = 4mh$, and $k \sin\theta = 2m \cos\theta$; hence (2) becomes

$$y^2 = \frac{4m}{\sin^2\theta}\, x = 4m'x, \quad \text{putting } \frac{4m}{\sin^2\theta} = 4m.$$

$$\text{Also, } \frac{4m}{\sin^2\theta} = 4m\,(1 + \cot^2\theta) = 4m\left(1 + \frac{k^2}{4m^2}\right) = 4\,(m + h) = SQ.$$

point of the parabola referred to QX', QY', is the equation required.

Let $AM (= h)$ $MQ (= k)$ be the co-ordinates of Q referred to AX, AY; then, since $TA = AM = h$ (by Art. 216), we have

$$\frac{TQ^2}{TA} = \frac{4h^2 + k^2}{h} = h + m, \quad \text{since } k^2 = 4mh$$
$$= SQ, \text{ by Art. 202.}$$

Hence $$m' = SQ.$$

It appears therefore that the equation of the parabola referred to QX' and QY' is of exactly the same form as the equation referred to AX and AY, and that the coefficient of x in both equations is four times the distance of the origin from the focus. The quantity $4m'$ is sometimes called the *parameter* of the diameter QX'.*

PROP. LXII.

233. A diameter and its tangent being taken as co-ordinate axes, to find the equation of the line touching the parabola at a given point (hk).

Proceeding exactly as in the note to Art. 214, we may shew that the equation of the tangent required is

$$ky - 2m'(x + h) = 0.$$

234. COR. 1. Hence the subtangent is, as before, twice the abscissa.

235. COR. 2. Hence, also, the equation of the line joining the points of contact of two tangents drawn from any point (hk) is, as before, $$ky - 2m'(x + h) = 0.$$

236. COR. 3. If in this equation we put $y = 0$, we find $x = -h$, and this is true whatever k be; hence, if QM (fig. 75) $= h$, and we take $QN = -h$, and draw NR parallel to QY'; the line joining the points of contact of two tangents drawn from any point of NR always passes through M. (See Art. 226.)

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* Suppose the double ordinate RR' to pass through the focus S; then, $PQTS$ being a parallelogram, we have $PQ = ST = SP$ (by Art. 217) $= m'$; hence by (1) we find (since $x = PQ$) $y^2 = 4m'^2$, and therefore $2y = 4m'$. Hence it appears that $4m'$ is equal to the double ordinate passing through the focus; *i.e.* the double ordinate which passes through the focus is equal to the parameter of the diameter of that ordinate.

237. Cor. 4. If we put $k = 0$, the equation becomes $x = -h$, which represents a line parallel to QY'. Hence the line joining the points of contact of two tangents drawn from any point of a diameter (produced of course) is parallel to the tangent at the extremity of that diameter, and is therefore bisected by that diameter.

This may be also seen immediately in the following manner. Let two tangents be drawn at the extremities of any double ordinate of a diameter, then the two subtangents (the diameter being axis of x) are equal, since they are each twice the abscissa of the double ordinate; and therefore the two tangents meet the diameter in the same point. Therefore, &c.

Various Problems connected with the Parabola.

238. A parabola being traced upon paper, to determine its axis.

Let fig. 76 be the given parabola; draw any two parallel chords PP' QQ', bisect them in M, N, draw RR' perpendicular to MN, bisect RR' in B, and draw BA parallel to MN; then AB is the axis required.

239. A parabola and its axis being given, to determine its focus.

Let fig. 77 be the given parabola and AX its axis; take any length AB on AX, draw BC equal to $2AB$ and perpendicular to AB, and through L, where AC produced meets the parabola, draw LS perpendicular to AX; then S is the focus required.

240. Having given S, SP in magnitude and position, and the tangent at P in position; to determine the parabola.

241. The circle described on SP as diameter touches the axis AY.

Let (hk) be the co-ordinates of P; then the equation of the circle, described on SP as diameter, is

$$\left(x - \frac{m+h}{2}\right)^2 + \left(y - \frac{k}{2}\right)^2 = \left(\frac{h-m}{2}\right)^2 + \left(\frac{k}{2}\right)^2,$$

or
$$x^2 - (m+h)\,x + y^2 - ky + mh = 0.$$

Put $x = 0$, and we find (since $k^2 = 4mh$)

$$y^2 - ky + \frac{k^2}{4} = 0,$$

which, being a perfect square, gives two equal values of y; therefore the circle touches the axis of y.

242. To describe a circle touching a parabola, and having its centre on the axis.

The equation of a circle, having its centre on the axis at a distance h from A, is $(x - h)^2 + y^2 = a^2$;

if (xy) be one of the points of intersection of this circle and the parabola, we have $y^2 = 4mx$; and therefore the equation of the circle becomes

$$(x - h)^2 + 4mx = a^2,$$

or

$$x^2 - 2(h - 2m)x + h^2 - a^2 = 0;$$

and if the circle touch the parabola, this ought to give two equal values of x; therefore we have

$$(h - 2m)^2 = h^2 - a^2,$$

or

$$a^2 = 4m(h - m),$$

which determines the radius of the circle in terms of h, to which we may give any value we please not less than m.

243. To shew, hence, that if d and d' be the diameters of the circles inscribed in, and circumscribed round, a portion of a parabola cut off by a double ordinate (fig. 78) whose length is q, the length of the corresponding abscissa being p; then

$$d + d' = p + q.$$

In the preceding problem, if we put $h = p - a$, we find

$$a^2 + 4ma + 4m^2 = 4mp = \frac{q^2}{4};$$

and therefore, putting $2a = d$, we have

$$d = q - 4m.$$

Again, let the equation of the circumscribed circle be

$$x^2 - 2a'x + y^2 = 0,$$

which, since it passes through the point $\left(p, \frac{q}{2}\right)$, gives

$$d' = 2a' = \frac{p^2 + \frac{q^2}{4}}{p} = p + 4m.$$

Hence $$d + d' = p + q.$$

244. If P, P' be the points of contact of two tangents drawn from a point T to a parabola, and if $T\dot{P}P' = 90°$; then TP is bisected by the directrix.

Let the co-ordinates of T be hk, and of P, $h'k'$; then the equations of TP and PP' are

$$k'y - 2m(x + h') = 0 \quad \ldots\ldots\ldots\ldots (1),$$
$$ky - 2m(x + h) = 0 \quad \ldots\ldots\ldots\ldots (2);$$

and therefore, since these lines are at right angles, we have

$$\frac{2m}{k} \cdot \frac{2m}{k'} = -1, \quad \text{or } kk' + 4m^2 = 0 \quad \ldots\ldots\ldots (3);$$

also, since (hk) is a point of (1), we have

$$k'k - 2m(h + h') = 0;$$

which, by (3), becomes

$$2m + (h + h') = 0, \quad \text{or } (-m) - h = h' - (-m);$$

which equation shews that the point of line TP, whose abscissa is $(-m)$, is half-way between the points T and P; *i.e.* TP is bisected by the directrix.

245. To find the equation of the parabola referred to the foot of the directrix (E) as origin, and the two tangents drawn from E (which, as we have seen, are at right angles to each other) as co-ordinate axes.

We make E the origin by putting $x - m$ for x, in the equation $y^2 = 4mx$, and we make the two tangents the axes of co-ordinates by putting, $x \cos(-45°) - y \sin(-45°)$, for x, and, $x \sin(-45°) + y \cos(-45°)$, for y; hence the equation of the parabola referred to the new axes, is

$$\left(\frac{x - y}{\sqrt{2}}\right)^2 = 4m\left(\frac{x + y}{\sqrt{2}} - m\right),$$

$$\text{or} \quad (x - y)^2 - 4m\sqrt{2}(x + y) + 8m^2 = 0;$$

which, putting $2m\sqrt{2} = a$ for brevity, may be reduced to the form

$$(x + y)^2 - 2a(x + y) + a^2 = 4xy,$$
$$\text{or} \quad x + y - a = \pm 2\sqrt{xy},$$
$$\text{or} \quad \sqrt{x} \pm \sqrt{y} = \sqrt{a},$$

which is the equation required in a remarkably simple form.

246. A parabola whose focus is given touches a given line, to determine the locus of its vertex.

Let S (fig. 79) be the given focus, and BC the given line; draw SQ perpendicular to BC, and let A be the vertex of the parabola. Then, SQ being a perpendicular from the focus upon a tangent, Q must be a point of the line drawn through the vertex perpendicular to the axis; therefore SAQ is a right angle. Hence A is always on the circumference of the circle described on SQ as diameter; which circle is therefore the locus required.

247. If TP (fig. 80) be any tangent of a parabola, $P'Q$ any other tangent meeting TP in Q; then $P'QS$ is an invariable angle, whatever be the position of $P'Q$.

Draw $AY'Y$ perpendicular to AS to meet the two tangents at Y' and Y. Then SYP and $SY'P'$ are right angles; therefore $SY'YQ$ is inscribable in a circle, and therefore $Y'QS = Y'YS$, which is an invariable angle whatever be the position of $P'Q$. Q. E. D.

248. If TQ, TQ', and QQ' be any three tangents of a parabola, the points T, Q, Q', and S, lie in the circumference of the same circle. (fig. 81).

By the last problem $SQQ' = SYY'$, and $SQ'Q = SY'Y$; and, since SYT and $SY'T$ are right angles, $SYY' + SY'Y = YTY'$; hence $SQQ' + SQ'Q = QTQ'$, which shews that Q, Q', T, and S, lie in the circumference of the same circle.

249. To draw a normal from a given point (ab) to a parabola.

The equation of the normal at the point (hk) is

$$2m(y-k)+k(x-h)=0;$$

and, if this line pass through the point (ab), we have

$$2m(b-k)+k(a-h)=0,$$

which, since $h = \dfrac{k^2}{4m}$, becomes

$$k^3 + (8m^2 - 4ma)k - 8m^2b = 0 \;\ldots\ldots\ldots\; (1),$$

which equation determines k; and then, since $h = \dfrac{k^2}{4m}$, we have h also.

Since (1) is a cubic equation, we may in general draw three different normals from a given point to a parabola.

250. To determine under what circumstances two of the three normals in the preceding problem coincide.

In such a case (1) must have two equal roots, and therefore, by the theory of equations,* we have

$$3k^2 + 8m^2 - 4ma = 0 \quad \ldots\ldots\ldots\ldots \quad (2),$$

which, putting $k^2 = 4mh$, gives

$$3h + 2m - a = 0.$$

Also, in virtue of (2), (1) becomes

$$k^3 + 4m^2b = 0;$$

hence we have
$$(4m^2b)^{\frac{2}{3}} = 4m\,\frac{a - 2m}{3};$$

which is the condition that must hold between a and b, in order that two of the three normals drawn from (ab) may coincide.

Hence every point of the curve whose equation is

$$y^{\frac{2}{3}} = p(x - 2m), \text{ where } p = \frac{1}{3}\left(\frac{4}{m}\right)^{\frac{1}{3}},$$

possesses the property, that two of the three normals, drawn from any point of it, coincide. This curve is called the *evolute* of the parabola, and sometimes the *semicubical parabola*. Its figure is shewn in (fig. 82.)

251. To find the lengths of the two tangents drawn from a given point (hk) to the parabola.

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* Let a and b be two roots of the equation

$$x^3 + px^2 + qx + r = 0 \quad \ldots\ldots\ldots \quad (3);$$

then we have $a^3 + pa^2 + qa + r = 0$, and $b^3 + pb^2 + qb + r = 0$; and therefore, subtracting and dividing by $a - b$, we find

$$a^2 + ab + b^2 + p\,(a + b) + q = 0 \quad \ldots\ldots \quad (4).$$

Now, suppose b to approach a in magnitude; then (4) will approach the form

$$3a^2 + 2pa + q = 0 \quad \ldots\ldots\ldots\ldots \quad (5);$$

and therefore, when b becomes equal to a, the condition (5) must hold. Hence, if two roots of (3) be equal to a, we have

$$a^3 + pa^2 + qa + r = 0,$$

$$\text{and} \quad 3a^2 + 2pa + q \quad = 0.$$

By Art. 210, we have the equation

$$r^2 - 2\,\frac{2m\cos\theta - k\sin\theta}{\sin^2\theta}\,.\,r + \frac{k^2 - 4mh}{\sin^2\theta} = 0,$$

and if the roots of this equation be equal, r is the length of a tangent drawn from (hk) to the parabola. Supposing then that this equation is a perfect square, we have

$$r = \frac{2m\cos\theta - k\sin\theta}{\sin^2\theta} \;\ldots\ldots (1) \qquad r^2 = \frac{k^2 - 4mh}{\sin^2\theta} \;\ldots\ldots (2).$$

Hence, if we put $k^2 - 4mh = c^2$, (2) gives

$$\sin\theta = \frac{c}{r} \qquad \cos\theta = \frac{\sqrt{(r^2 - c^2)}}{r};$$

and therefore (1) $\times \sin^2\theta$ becomes, by (2),

$$c^2 = 2m\sqrt{(r^2 - c^2)} - kc,$$

$$\text{and} \quad \therefore\; r^2 = c^2\left\{\left(\frac{c+k}{2m}\right)^2 + 1\right\},$$

which is the expression for the squares of the two required lengths, the value of c being $\pm\sqrt{(k^2 - 4mh)}$.

252. If T be the point of intersection of two tangents drawn at the points P and P' of the parabola; to shew that the distance of T from the axis is an arithmetic mean between the distanecs of P and P' from the axis.

Let T be (hk) and P, or P', (xy), then

$$ky = 2m(x + h),$$

which, since $y^2 = 4mx$, becomes

$$y^2 - 2ky + 4mh = 0.$$

Let y and y' be the two roots of this, then

$$y + y' = 2k,$$

which shews that k is an arithmetic mean between y and y'. Q. E. D.

CHAPTER IX.

OF THE ELLIPSE. PROPERTIES OF THE ELLIPSE NOT CONNECTED WITH TANGENTS OR DIAMETERS. OF THE ECCENTRIC ANGLE. PROPERTIES CONNECTED WITH TANGENTS. PROPERTIES CONNECTED WITH DIAMETERS. VARIOUS PROBLEMS.

253. The equation of the ellipse referred to its centre (C) as origin is, by Chap. VI.,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

a being the semi-axis major (CA), and b the semi-axis minor (CB).

The polar equation referred to the focus (S) is, by Article 177,

$$r=\frac{l}{1-e\cos\theta},$$

l being the semi-latus-rectum, and e the eccentricity.

We have also, by Art. 176,

$$SC=ae,\quad EC=\frac{a}{e},\quad b^2=a^2(1-e^2),\quad l=a(1-e^2)=\frac{b^2}{a}.$$

The curve is perfectly symmetrical with respect to AA' and BB', so that there is another focus S' and another directrix $E'K'$ (fig. 57), corresponding to S and EK, CS' and CE' being respectively ae and $\frac{a}{e}$.

We shall now, by means of these equations and formulæ, investigate the various remarkable properties of the ellipse.

Properties of the Ellipse not connected with Tangents or Diameters.

Prop. LXIII.

254. To find the distances of any point of an ellipse from the two foci.

Let (xy) be any point (P) of the ellipse; then, since ae and o are the co-ordinates of S', we have

$$S'P^2 = (x - ae)^2 + y^2,$$

which, since $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = (a^2 - x^2)(1 - e^2)$, becomes

$$S'P^2 = e^2x^2 - 2aex + a^2 = (a - ex)^2,$$

therefore $$S'P = \pm(a - ex).$$

$S'P$ here means the absolute distance of P from S' without regard to sign; hence, since e is < 1 and $x < a$, and therefore $a - ex$ a positive quantity, we must reject the lower sign. We have therefore $$S'P = a - ex \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In like manner we find, since $-ae$ and o are the co-ordinates of S,

$$SP^2 = (x + ae)^2 + y^2$$
$$= e^2x^2 + 2aex + a^2;$$

and therefore $$SP = a + ex \ldots\ldots\ldots\ldots\ldots (2),*$$

(1) and (2) are the expressions for the required distances.

255. Cor. 1. By adding (1) and (2) we find immediately

$$SP + S'P = 2a.$$

Hence the sum of the distances of any point of an ellipse from the two foci, is always equal to the axis major.†

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* (1) and (2) may be readily deduced from the polar equation of the ellipse, and geometrically as follows.

In fig. 83, drawing QPQ' parallel to AA', we have

$$SP = eQP = e(EC + CM)$$
$$= a + ex, \quad \text{since } EC = \frac{a}{e}.$$

Also, $$S'P = eQ'P = e(E'C - CM)$$
$$= a - ex, \quad \text{since } E'C = \frac{a}{e}.$$

We may also prove geometrically, that $SP + S'P = 2a$, as follows:

$$SP + S'P = e(QP + Q'P) = e(EC + E'C) = 2a.$$

† Conversely. To find the locus of a point P (fig. 84), the sum of whose distances from two fixed points, S and S', is invariable.

Let $SP = r$, $S'P = r'$, $PSS' = \theta$, $SS' = 2c$; and let $2a$ be the invariable sum of SP and $S'P$. Then we have $r + r' = 2a$, or

$$r'^2 = r^2 + 4a^2 - 4ar,$$

but $$r'^2 = r^2 + 4c^2 - 4cr\cos\theta;$$

256. Cor. 2. This remarkable property suggests the following method of drawing an ellipse mechanically.

Fix the extremities of a string SPS' at two points S and S', (fig. 84), and with a pencil point P, keeping the string always full stretched, describe the curve $PAP'A'$; then this curve will be an ellipse having its foci at S and S', and its axis major equal to the length of the string.

257. Cor. 3. We may shew, as in Art. 203, that, to whatever axes of co-ordinates the ellipse be referred, the distance of any point of it from either focus is expressed by a *linear function* of the co-ordinates of that point.

258. Cor. 4. We may obtain the values of SP and $S'P$ very easily from Art. 204. For the distance (r) of any point (xy) of the ellipse from either focus must be a linear function of x and y, which gives
$$r = Ax + By + C,$$
and the equation of the directrix must be
$$Ax + By + C = 0,$$
which gives
$$B = 0, \quad C = \pm A\frac{a}{e}.$$

Also, since $\sqrt{(A^2 + E^2)}$ = the eccentricity, we have $A = \pm e$; hence
$$r = \pm (a \pm ex),$$
$$= a \pm ex,$$
since we suppose r to be a positive quantity.

Prop. LXIV.

259. If PSP' be any chord drawn through S, the semi-latus-rectum is a harmonic mean between SP and SP'.

This may be shewn exactly in the same way as in Art. 207.

Prop. LXV.

260. Two given lines being drawn through the vertex A',

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therefore $a^2 - ar = c^2 - cr\cos\theta$, or $r = \dfrac{a^2 - c^2}{a - c\cos\theta}$,
which, if we put $c = ae$, becomes
$$r = \frac{a(1-e^2)}{1 - e\cos\theta},$$
which represents an ellipse.

to find the equation of the line joining the other two points in which they intersect the ellipse.

Let the equations of the two lines be

$$y = \alpha(x - a), \quad y = \alpha'(x - a).$$

Then both lines together are represented by the equation

$$y^2 - (\alpha + \alpha')(x - a)y + \alpha\alpha'(x - a)^2 \ldots\ldots (1).$$

Now where this locus intersects the ellipse, we have

$$y^2 = -\frac{b^2}{a^2}(x^2 - a^2).$$

Substituting this value of y^2 in (1), and dividing out $x-a$, we find

$$-\frac{b^2}{a^2}(x + a) - (\alpha + \alpha')y + \alpha\alpha'(x - a) = 0 \ldots . (2),$$

which, as in Art. 208, is the equation required.

Putting $y = 0$ in (2), we find

$$-\frac{b^2}{a^2}(x + a) + \alpha\alpha'(x - a) = 0,$$

which, if $\alpha\alpha' = -1$, gives $\quad x = \dfrac{a^2 - b^2}{a^2 + b^2} \cdot a.$

Hence, if any two lines be drawn from the vertex of an ellipse at right angles to each other, the line joining the other two points in which they cut the ellipse always passes through the same point of the major axis.

Prop. LXVI.

261. If a circle be described on the axis major as diameter, and the ordinate MP be produced to meet it in P'; then $MP : MP' :: b : a$.

For from the equation of the ellipse we have

$$MP^2 = \frac{b^2}{a^2}(a^2 - x^2);$$

and from the equation of the circle

$$MP'^2 = a^2 - x^2,$$

therefore $\quad \dfrac{MP}{MP'} = \dfrac{b}{a}. \quad$ Q. E. D.

PROP. LXVII.

262. To determine the length (r) of a line drawn, at an angle θ to the axis of x, from any point (hk) to the ellipse.

Proceeding exactly as in Arts. 128, 210, putting $h + r\cos\theta$ and $k + r\sin\theta$ for x and y, in the equation of the ellipse, we find

$$\frac{(h + r\cos\theta)^2}{a^2} + \frac{(k + r\sin\theta)^2}{b^2} = 1,$$

which may be put in the form

$$Ur^2 + 2Vr + W = 0 \text{ .... } (1),$$

where $\quad U = \dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}, \quad V = \dfrac{h\cos\theta}{a^2} + \dfrac{k\sin\theta}{b^2},$

$$W = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1.$$

(1) determines the required distance. Since it is a quadratic equation, the right line must in general intersect the ellipse in two points, whose distances from (hk) are the roots of (1).

263. COR. 1. Using the same notation as in Art. 129, we have

$$PQ + PQ' = -2\frac{V}{U} \text{ .... } (2),$$

$$PQ \,.\, PQ' = \frac{W}{U} \text{ .... } (3).$$

264. COR. 2. Hence, as in Art. 132, if (hk) be the middle point of QQ', we have $V = 0$, or

$$\frac{h\cos\theta}{a^2} + \frac{k\sin\theta}{b^2} = 0 \text{ .... } (4);$$

which is therefore the condition necessary in order that (hk) may be the middle point of the chord which makes an angle θ with the axis of x.

PROP. LXVIII.

265. If QQ' and RR' be two chords of an ellipse, and P their point of intersection, the ratio $PQ.PQ' : PR.PR'$ is not altered by moving each chord parallel to itself, and so shifting the position of P in any manner.

By the previous Article we have

$$PQ \,.\, PQ' = \frac{W}{U} = \frac{b^2h^2 + a^2k^2 - a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta},$$

θ being the angle which QQ' makes with the axis, and (hk) the point P. In the same way, if ϕ be the angle which RR' makes with the axis, we have

$$PR \,.\, PR' = \frac{b^2h^2 + a^2k^2 - a^2b^2}{b^2 \cos^2 \phi + a^2 \sin^2 \phi};$$

therefore
$$\frac{PQ \,.\, PQ'}{PR \,.\, PR'} = \frac{b^2 \cos^2 \phi + a^2 \sin^2 \phi}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}.$$

Now if we move each chord parallel to itself in any manner, we do not alter θ or ϕ; hence the truth of the proposition is manifest.

Prop. LXIX.

266. To find the polar equation of the ellipse referred to the centre as pole.

In the equation
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

put $x = r \cos \theta$, $y = r \sin \theta$, and it becomes

$$r^2 \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}\right) = 1 \quad \ldots\ldots\ldots\ldots \quad (1),$$

or
$$r = \frac{ab}{\sqrt{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)}} \quad \ldots\ldots\ldots\ldots \quad (2),$$

which is the equation required. It is sometimes given in the form
$$r = \frac{a\sqrt{(1 - e^2)}}{\sqrt{(1 - e^2 \cos^2 \theta)}}.$$

267. Cor. By (1) we have
$$\frac{1}{r^2} = \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2};$$

and if r' be the radius vector drawn at right angles to r, we have, putting $\theta + \frac{\pi}{2}$ for θ,

$$\frac{1}{r'^2} = \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2}.$$

Hence, by addition, we find
$$\frac{1}{r^2} + \frac{1}{r'^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

It appears therefore, that the sum of the reciprocal squares of two radii vectores, drawn from the centre of an ellipse at right angles to each other, is invariable.

Of the Eccentric Angle in the Ellipse.

PROP. LXX.

268. The co-ordinates x and y of any point of an ellipse may be put in the form $x = a \cos\phi$, $y = b \sin\phi$.

Let us assume, as we may, that $\frac{x}{a} = \cos\phi$; *i.e.* let ϕ be the angle whose cosine is $\frac{x}{a}$; then, substituting in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we find
$$\frac{y^2}{b^2} = 1 - \cos^2\phi = \sin^2\phi,$$

and therefore
$$\frac{y}{b} = \sin\phi.$$

Hence we may assume $x = a \cos\phi$, and $y = b \sin\phi$.

269. COR. 1. If we describe a circle $AP'A'$ (fig. 85) on AA' as diameter, produce the ordinate MP to meet it in P', and join P' and C; $P'CM$ is the angle whose cosine is $\frac{x}{a}$.

$$\text{For}\quad \cos P'CM = \frac{CM}{CP'} = \frac{CM}{CA'} = \frac{x}{a}.$$

Hence, if we produce the ordinate to meet a circle described on the axis major as diameter, and draw a line from the point of meeting to the centre, ϕ is the angle which that line makes with the axis major.

270. COR. 2. Draw PQR parallel to $P'C$ to meet CB' in R and CA in Q; then $PQ = P'C = a$, and $PQA' = P'CA' = \phi$.

Hence ϕ is the angle which a line, equal to a, drawn from P to BB', makes with AA'.

Since $y = b \sin\phi$, we have

$$b \sin\phi = PM = PQ \sin PQA' = PQ \sin\phi,$$

therefore
$$PQ = b.$$

Hence ϕ is the angle which a line, equal to b, drawn from P to AA', makes with AA'.

Hence, if a line be drawn from any point P of an ellipse to BB', equal in length to a, the part of it intercepted between P and AA' is equal to b.

271. Cor. 3. Hence, if AA', BB' be two lines at right angles to each other (fig. 86), and PQR any line intersecting them, so that $PR = a$, $PQ = b$; then P is a point of the ellipse whose axes coincide with AA' and BB', and are equal to $2a$ and $2b$.

272. Cor. 4. Hence, if we suppose PQR to be a ruler having a pin at R and another at Q, and if AA', BB' be grooves in which these pins run, a pencil at the point P will trace an ellipse. In this manner an instrument for drawing ellipses, called an *elliptic compass*, has been constructed.

273. The angle ϕ we have here introduced is of considerable importance, as will appear, in proving several properties of the ellipse. It is made use of in Astronomy, and has been termed the *eccentric anomaly*, certain angles being called anomalies by astronomers. Instead of the word anomaly we shall employ the word angle, and accordingly call ϕ the *eccentric angle of the point P*.

If therefore ϕ be the eccentric angle of any point of an ellipse, $a \cos \phi$ and $b \sin \phi$ are the co-ordinates of that point.

Properties of the Ellipse connected with Tangents.

Prop. LXXI.

274. To find the angle which the line touching an ellipse at any point (hk) makes with the axis of x.

As in Art. 135, if we suppose Q and Q' in Art. 263, to coincide, the line PQQ' becomes a tangent at the point Q; and we then have, by equation (2),

$$PQ = -\frac{V}{U};$$

which, if we suppose P to coincide with Q, gives $V = 0$, or

$$\frac{h\cos\theta}{a^2} + \frac{k\sin\theta}{b^2} = 0;$$

and therefore $$\tan\theta = -\frac{b^2h}{a^2k}.$$

Hence $-\frac{b^2h}{a^2k}$ is the tangent of the angle which the line touching the ellipse at the point (hk) makes with the axis of x.

275. COR. 1. Hence the equation of the tangent at (hk) is, by Art. 60,

$$\frac{h}{a^2}(x-h) + \frac{k}{b^2}(y-k) = 0,$$

which, since $\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$, becomes

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).*$$

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* This result may also be obtained, as in the case of the parabola, as follows.

The equation of the line joining any two points (hk), $(h'k')$ of the ellipse is

$$y - k = \frac{k'-k}{h'-h}(x-h) \quad\ldots\ldots\ldots (4).$$

Now, subtracting the equations

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1, \qquad \frac{h'^2}{a^2} + \frac{k'^2}{b^2} = 1,$$

we find $\frac{k'^2-k^2}{b^2} = -\frac{h'^2-h^2}{a^2}$, and $\therefore \frac{k'-k}{h'-h} = -\frac{b^2}{a^2}\,\frac{h'+h}{k'+k}$,

and therefore (4) becomes

$$y - k = -\frac{b^2}{a^2}\,\frac{h'+h}{k'+k}(x-h);$$

which equation may be made to differ from

$$y - k = -\frac{b^2}{a^2}\,\frac{h}{k}(x-h) \quad\ldots\ldots\ldots (5)$$

as little as we please, by bringing the point $(h'k')$ sufficiently near to (hk). Hence, reasoning exactly as in the case of the parabola, (5) must be the tangent at (hk). (5) may be reduced, as in the text, to the form

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1.$$

276. Cor. 2. Hence, if T and T' be the points where the tangent at (hk) meets the major and minor axes, we have, by Art. 57,

$$CT = \frac{a^2}{h} \qquad CT' = \frac{b^2}{k}.*$$

Prop. LXXII.

277. If $(ax + \beta y = \gamma)$ be the equation of any tangent to the ellipse, to determine what relation must hold between a, β, and γ.

Let (hk) be the point of contact; then the two equations

$$\frac{ax}{\gamma} + \frac{\beta y}{\gamma} = 1, \quad \text{and} \quad \frac{hx}{a^2} + \frac{ky}{b^2} = 1,$$

must represent the same line. Therefore

$$\frac{a}{\gamma} = \frac{h}{a^2}, \qquad \frac{\beta}{\gamma} = \frac{k}{b^2};$$

and therefore, since $\left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 = 1$, we have

$$\left(\frac{aa}{\gamma}\right)^2 + \left(\frac{b\beta}{\gamma}\right)^2 = 1.$$

Hence the required relation between a, β, and γ is

$$\gamma^2 = a^2a^2 + b^2\beta^2.\dagger$$

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* This result may be proved geometrically as follows.

On AA' (fig. 87) as diameter describe a circle, produce any two ordinates, MP and $M'P'$, to meet it in Q and Q', and draw the lines $Q'Q$, $P'P$; then these lines produced must meet CX in the same point T, as is evident from the fact, that $MP : MQ = M'P' : M'Q'$, by Art. 261. Now if P' and Q' be supposed to approach P and Q, PT and QT will ultimately become tangents at P and Q. But, if QT be a tangent at Q, CQT is a right angle, and therefore $CT : CQ = CQ : CM$; which, since $CQ = a$, gives $CT = \frac{a^2}{h}$. And in the same way we may shew that $CT' = \frac{b^2}{k}$.

† This result may be obtained independently in the following manner.

Let (xy) be either of the points of intersection of the line and ellipse represented by

$$ax + \beta y = \gamma \ldots\ldots (3), \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots (4).$$

Then $(3)^2 - (4)\gamma^2$ gives $\left(a^2 - \frac{\gamma^2}{a^2}\right)x^2 + 2a\beta xy + \left(\beta^2 - \frac{\gamma^2}{b^2}\right)y^2 = 0 \ \therefore \ldots (5).$

278. If $y = \alpha x + \gamma$ be the equation of a tangent, this relation gives $\gamma^2 = a^2\alpha^2 + b^2$: hence the equation of any tangent of an ellipse may be put in the form

$$y = \alpha x \pm \sqrt{(a^2\alpha^2 + b^2)}.$$

The double sign shews that *two* tangents may be drawn at the same angle to the axis of x; which is manifestly true.

PROP. LXXIII.

279. To find the equation of the normal at any point (hk) of an ellipse, and the portions it cuts off from the axes.

The tangent makes an angle $\tan^{-1}\left(-\dfrac{b^2h}{a^2k}\right)$ with the axis of x, and the normal is the perpendicular to it drawn from the point (hk); therefore the equation of the normal is

$$a^2k\,(x - h) - b^2h\,(y - k) = 0,$$

or

$$\frac{a^2}{h}\,x - \frac{b^2}{k}\,y = a^2 - b^2,$$

which is the equation required.

Hence, if G and G' be the points where the normal meets the axis major and the axis minor, we have

$$CG = \frac{a^2 - b^2}{a^2}\,h = e^2h,$$

$$CG' = \frac{b^2 - a^2}{b^2}\,k = -\frac{a^2}{b^2}\,e^2k,$$

which are the portions required.

280. COR. 1. Hence the normal at P bisects the angle SPS'.

For we have (in fig. 88)

$$SG = SC + CG = ae + e^2h = eSP, \text{ by Art. 254},$$

$$S'G = S'C - CG = ae - e^2h = eS'P;$$

This equation gives, in general, two values of $\dfrac{y}{x}$; but, if (3) be a tangent, the two values ought to become equal, and therefore (5) ought to be a perfect square; which gives

$$\left(\alpha^2 - \frac{\gamma^2}{a^2}\right)\left(\beta^2 - \frac{\gamma^2}{b^2}\right) = \alpha^2\beta^2, \quad \text{or } \gamma^2 = a^2\alpha^2 + b^2\beta^2, \text{ as above.}$$

therefore $SG : S'G = SP : S'P$,

which shews that PG bisects the angle SPS'. Q. E. D.*

281. Cor. 2. Hence the tangent at P makes equal angles with SP and $S'P$. This result may be proved geometrically by means of the property, $SP + S'P = 2a$, in exactly the same way as in the case of the parabola. See Art. 217, note.

Prop. LXXIV.

282. To find the locus of the intersection of a tangent with a perpendicular let fall upon it from either focus.

By Art. 277 the equation of the tangent is

$$\alpha x + \beta y = \pm \sqrt{(a^2\alpha^2 + b^2\beta^2)} \quad \dots\dots\dots\dots (1),$$

and the equation of a line drawn perpendicular to this from S or S', is $\beta(x \pm ae) - \alpha y = 0$, or

$$\beta x - \alpha y = \pm \beta \sqrt{(a^2 - b^2)} \quad \dots\dots\dots\dots (2):$$

$(1)^2 + (2)^2$ gives immediately, dividing out $\alpha^2 + \beta^2$,

$$x^2 + y^2 = a^2.$$

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* The following is a more analytical proof of this property.

Let θ and θ' be the angles which SP and $S'P$ make with AA'; then we have

$$\tan\theta = \frac{k}{h + ae}, \qquad \tan\theta' = \frac{k}{h - ae},$$

and therefore $\tan(\theta + \theta') = \dfrac{2hk}{h^2 - a^2e^2 - k^2}$;

which, if we put $h = a\cos\phi$, and therefore $k = b\sin\phi$ (Art. 268), becomes

$$\tan(\theta + \theta') = \frac{2ab\sin\phi\cos\phi}{a^2\cos^2\phi - a^2 + b^2 - b^2\sin^2\phi}$$

$$= \frac{2ab\sin\phi\cos\phi}{b^2\cos^2\phi - a^2\sin^2\phi}.$$

Now, if ψ be the angle which PG makes with AA', we have

$$\tan\psi = \frac{a^2h}{b^2k} = \frac{a\sin\phi}{b\cos\phi},$$

and therefore $\tan 2\psi = \dfrac{2ab\sin\phi\cos\phi}{b^2\cos^2\phi - a^2\sin^2\phi}$.

Hence $\theta + \theta' = 2\psi$; which shews that PG bisects the angle SPS'.

Hence the locus required is a circle described on the axis major as diameter.*

283. Cor. This proposition suggests the following method of drawing (geometrically) a tangent, or rather two tangents, to an ellipse from any given point U (fig. 90).

Join S and U, and on SU, as diameter, describe a circle cutting the circle described on AA', as diameter, at R and R'; then the lines drawn from U through R and R' will touch the ellipse.

This is evident, since SRU and $SR'U$ are right angles, and therefore, by the proposition just proved, the lines drawn from U through R and R' must be tangents to the ellipse.

Prop. LXXV.

284. To determine analytically the tangents of an ellipse which pass through any given point.

The equation of the tangent is

$$y - \alpha x = \pm\sqrt{(a^2\alpha^2 + b^2)}.$$

Suppose it to pass through a given point (hk), then we have

$$(k - \alpha h)^2 = a^2\alpha^2 + b^2,$$

or
$$\alpha^2 + \frac{2hk}{a^2 - h^2}\,\alpha + \frac{b^2 - k^2}{a^2 - h^3} = 0 \;\ldots\ldots\ldots (1),$$

which equation gives two values of α; let them be α and α'; then the equations of the tangents required are

$$(y - k) = \alpha\,(x - h)\ldots(2),\quad y - k = \alpha'\,(x - h)\ldots(3).$$

285. Cor. To determine the locus of the point of intersection of two tangents at right angles to each other.

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* This may be shewn geometrically as follows:

Let SY be the perpendicular from S upon the tangent (YP) at P (fig. 89); produce $S'P$ and SY to meet in Q. Then, since $QPY = SPY$, by Art. 281, we have $PQ = PS$, and therefore, by Art. 255, $S'Q = 2a$; and, since $SC = CS'$ and $SY = YQ$, we have $CY = \frac{1}{2}\,S'Q = a$. Therefore Y is a point on the circumference of the circle described on the major axis as diameter.

If (2) and (3) be at right angles, we have

$$aa' + 1 = 0;$$

but, by (1), $$aa' = \frac{b^2 - k^2}{a^2 - h^2},$$

therefore $$\frac{b^2 - k^2}{a^2 - h^2} + 1 = 0.$$

or $$h^2 + k^2 = a^2 + b^2.$$

Hence the locus of the point (hk) is a circle described with C as centre and $\sqrt{(a^2 + b^2)}$ as radius.

This may also be shewn very readily as follows.

The equation of any tangent is

$$\alpha x + \beta y = \pm \sqrt{(a^2\alpha^2 + b^2\beta^2)}\ldots\ldots\ldots\ldots(1),$$

and the equation of a tangent at right angles to it is

$$\beta x - \alpha y = \pm \sqrt{(a^2\beta^2 + b^2\alpha^2)}\ldots\ldots\ldots\ldots(2),$$

$(1)^2 + (2)^2$ gives immediately

$$x^2 + y^2 = a^2 + b^2,$$

which is the same result as before.

PROP. LXXVI.

286. If p and p' be the perpendiculars from S and S' upon any tangent, to shew that $pp' = b^2$.

The equation of the tangent being

$$\alpha x + \beta y = \gamma = \sqrt{(a^2\alpha^2 + b^2\beta^2)},$$

we have, by Art. 69, the points S and S' being $(-ae, 0)$ and $(ae, 0)$,

$$p = \frac{\gamma + \alpha ae}{\sqrt{(\alpha^2 + \beta^2)}}, \quad p' = \frac{\gamma - \alpha ae}{\sqrt{(\alpha^2 + \beta^2)}};$$

therefore $$pp' = \frac{\gamma^2 - \alpha^2(a^2 - b^2)}{\alpha^2 + \beta^2}$$

$$= \frac{b^2\beta^2 + b^2\alpha^2}{\alpha^2 + \beta^2} = b^2. \quad \text{Q. E. D.}$$

287. COR. If we denote SP and $S'P$ by r and r', it is evident, since SP and $S'P$ make equal angles with the tangent at P, that

$$\frac{p}{p'} = \frac{r}{r'};$$

multiplying this by $pp' = b^2$, we find

$$p^2 = b^2 \frac{r}{r'}$$

$$= b^2 \frac{r}{2a - r}, \quad \text{since } r' + r = 2a,$$

which is an expression for the perpendicular from S upon the tangent at P, in terms of SP.

PROP. LXXVII.

288. To find the perpendicular from C upon the tangent, in terms of the angle which the tangent makes with the axis of x.

The equation of a line making an angle θ with the axis of x and at a perpendicular distance p from the origin is, by Art. 57,

$$y \cos \theta - x \sin \theta = p,$$

and, by Art. 277, if this be a tangent, we have

$$p^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta,$$

which is therefore the expression for the perpendicular required.

It is often put in the form

$$p^2 = a^2 (1 - e^2 \cos^2 \theta),$$

by substituting for b^2 its value $a^2 - a^2e^2$.

PROP. LXXVIII.

289. To determine the equation of the line joining the points of contact of the two tangents drawn from a given point to an ellipse.

Let (hk) be the given point, and (xy) either of the points of contact; then the equation of the tangent gives

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1;$$

and, reasoning just as in Art. 139, we may shew that this is the equation of the line joining the two points of contact.

290. COR. 1. Hence, reasoning exactly as in Art. 140, we may shew that the locus of the intersection of the two tangents, drawn at the extremities of any chord passing through (hk), is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1.$$

291. Cor. 2. Hence the locus of the intersection of the two tangents, drawn at the extremities of any chord passing through S, is the directrix EK. For, putting $k = 0$, $h = -ae$, the equation of the locus becomes $x = -\frac{a}{e}$, which represents EK.

Properties of the Ellipse connected with Diameters.

Prop. LXXIX.

292. To determine the diameter of a given system of parallel chords in an ellipse.

Let θ be the angle which any chord makes with the axis of x, and (xy) its middle point; then, by Art. 264, we have

$$\frac{x \cos\theta}{a^2} + \frac{y \sin\theta}{b^2} = 0 \quad \ldots\ldots\ldots\ldots (1),$$

and consequently, if we suppose the chord to move parallel to itself, and therefore θ to remain invariable, this equation represents the locus of its middle point. Hence (1) is the equation of the diameter of the system of parallel chords which make an angle θ with the axis of x.

293. Cor. 1. If we suppose the chord to move parallel to itself until its extremities coincide, it then becomes a tangent; and hence it follows, that the tangent at the extremity of any diameter is parallel to the chords of that diameter. This may also be easily seen from the equations of the diameter and tangent; for, if (hk) be the extremity of any diameter, and θ the angle its chords make with the axis of x, we have, by the equation of the diameter just obtained,

$$\tan\theta = -\frac{b^2 h}{a^2 k};$$

hence, by Art. 274, θ is also the angle which the tangent at (hk) makes with the axis of x. Therefore the tangent is parallel to the chords.

294. Cor. 2. The equation (1) shews that every diameter of the ellipse passes through the centre (C). Therefore every chord drawn through C is bisected at C.

295. Cor. 3. If θ' be the angle which the diameter (1) makes with the axis of x, we have

$$\tan\theta' = -\frac{b^2\cos\theta}{a^2\sin\theta}, \quad \text{and therefore} \quad \tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

Hence, if θ and θ' be the angles which a system of parallel chords and their diameter respectively make with the axis of x, we have the relation

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}\ldots\ldots\ldots\ldots\ldots(2),$$

from which we may find either θ' in terms of θ, or θ in terms of θ'.

Prop. LXXX.

296. If one diameter be parallel to the chords of another, the latter diameter will also be parallel to the chords of the former.

Let us denote the two diameters by D and D_1, let θ, θ_1 be the angles which their chords, and θ', θ_1' the angles which they themselves, respectively, make with the axis of x. Then, by (2), we have

$$\tan\theta\tan\theta' = \tan\theta_1\tan\theta_1',$$

which shews that, if $\tan\theta = \tan\theta_1'$, then $\tan\theta_1 = \tan\theta'$. Hence, if the chords of D be parallel to D_1, the chords of D_1 will also be parallel to D. Q. E. D.

Two diameters thus related are called *conjugate diameters*.

297. Cor. 1. Hence, if θ and θ' be the angles which two conjugate diameters make with the axis major, we have

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

298. Cor. 2. If (xy) and $(x'y')$ be any points on the two conjugate diameters respectively, and therefore $\frac{y}{x} = \tan\theta$, $\frac{y'}{x'} = \tan\theta'$; we have

$$\frac{y}{x}\frac{y'}{x'} = -\frac{b^2}{a^2} \quad \text{or} \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$

Prop. LXXXI.

299. If ϕ and ϕ' be the eccentric angles of the extremities of two conjugate diameters, then $\phi' = \phi + 90°$.

Let (xy) and $(x'y')$ be the extremities P and D of two conjugate diameters CP and CD (fig. 91), and ϕ and ϕ' the corresponding eccentric angles, then

$$x = a \cos\phi, \quad y = b \sin\phi, \quad x' = a\cos\phi', \quad y' = b\sin\phi',$$

and therefore, by the preceding Article, we have

$$\cos\phi\cos\phi' + \sin\phi\sin\phi' = 0, \quad \text{or} \quad \cos(\phi' - \phi) = 0,$$

which shews that $\phi' - \phi = 90°$. Q. E. D.*

This property of conjugate diameters, with reference to the eccentric angle, will be found of great use in all problems relating to conjugate diameters.

300. Cor. Hence

$$x' = a\cos\left(\phi + \frac{\pi}{2}\right) = -a\sin\phi = -\frac{a}{b}y,$$

and $$y' = b\sin\left(\phi + \frac{\pi}{2}\right) = b\cos\phi = \frac{b}{a}x,$$

which formulæ determine the co-ordinates of D in terms of those of P.

Prop. LXXXII.

301. The sum of the squares of CP and CD, and the area of the parallelogram completed upon CP and CD, are invariable.

Let $CP = r$, $CD = r'$, $PCA = \theta$, $DCA = \theta'$, (fig. 91). Then

$$r^2 = x^2 + y^2 = a^2\cos^2\phi + b^2\sin^2\phi,$$

and, putting $\phi + \frac{\pi}{2}$ for ϕ, we change r into r'; therefore

$$r'^2 = a^2\sin^2\phi + b^2\cos^2\phi.$$

Hence $$r^2 + r'^2 = a^2 + b^2 \quad \text{.............. (1).}$$

---

* Hence the following simple construction for determining two conjugate diameters of an ellipse.

On AA' (fig. 92) as diameter describe a circle $A'P'D'A$, from C draw any two lines CP', CD' at right angles to each other, and draw the ordinates $P'PM$, $D'DN$; then CP and CD are conjugate diameters.

Again, if A be the area of the parallelogram completed upon CP and CD, we have

$$A = rr' \sin(\theta' - \theta) = rr' (\sin\theta' \cos\theta - \cos\theta' \sin\theta)$$

$$= xy' - x'y$$

$$= ab \sin(\phi' - \phi).$$

Hence, since $\phi' - \phi = \frac{\pi}{2}$, we have

$$A = ab \quad\quad (2).$$

(1) shews that the sum of the squares of CP and CD is invariable, and (2) shews that the area of the parallelogram completed upon CP and CD is invariable. Q. E. D.

302. Cor. 1. Since the tangents at P and D are respectively parallel to CD and CP, it follows that the area of every parallelogram circumscribing an ellipse, having its sides parallel to two conjugate diameters, is the same.

303. Cor. 2. In the above expressions for r^2 and r'^2, putting $1 - \cos^2\phi$ for $\sin^2\phi$, we have

$$r^2 = b^2 + a^2e^2\cos^2\phi = b^2 + e^2x^2,$$

$$r'^2 = a^2 - a^2e^2\cos^2\phi = a^2 - e^2x^2.$$

Hence, since, by Art. 254, $SP = a + ex$, $S'P = a - ex$, we have $SP \cdot S'P = r'^2$, or

$$SP \cdot S'P = CD^2.$$

Prop. LXXXIII.

304. To find the equation of the ellipse referred to any two conjugate diameters, CP and CD, as axes of co-ordinates (fig. 93).

Draw any chord QMQ' parallel to DCD'; assume $CM = x$, $MQ = y$, $CP = a'$, $CD = b'$. Then, by Art. 265, we have

$$\frac{MP \cdot MP'}{CP \cdot CP'} = \frac{MQ \cdot MQ'}{CD \cdot CD'}.$$

But $CP = CP' = a'$, $MP = a' - x$, $MP' = a' + x$, $CD = CD' = b'$, and $MQ = MQ' = y$; therefore

$$\frac{a'^2 - x^2}{a'^2} = \frac{y^2}{b'^2},$$

or $$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1 \text{ .... } (1),$$

which is the equation required.*

Hence it appears that the equation of the ellipse referred to any two conjugate diameters is of exactly the same form as that referred to CA and CB.

PROP. LXXXIV.

305. The ellipse being referred to two conjugate diameters, to find the equation of the line touching it at any proposed point (hk).

Proceeding exactly as in the notes to Arts. 214 and 275, we may shew that the equation of the tangent required is

$$\frac{hx}{a'^2} + \frac{ky}{b'^2} = 1.$$

---

* This result may be obtained by transformation of co-ordinates in the following manner.

If, in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (which is referred to CA and CB), we put $x \cos\theta + y \cos\theta'$ for x, and $x \sin\theta + y \sin\theta'$ for y, it becomes

$$\frac{(x\cos\theta + y\cos\theta')^2}{a^2} + \frac{(x\sin\theta + y\sin\theta')^2}{b^2} = 1 \ldots (2);$$

and this substitution amounts to turning the axes of co-ordinates round the origin, until the new axes of x and y make angles θ and θ' with the old axis of x. (See Art. 110.)

Now, θ and θ' being disposable, we may assume that

$$\frac{\cos\theta\cos\theta'}{a^2} + \frac{\sin\theta\sin\theta'}{b^2} = 1,$$

(which amounts to supposing the new axes to be conjugate diameters); and then (2) becomes

$$\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)x^2 + \left(\frac{\cos^2\theta'}{a^2} + \frac{\sin^2\theta'}{b^2}\right)y^2 = 1 \ldots (3).$$

But, by Art. 266, we have

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{r^2}, \qquad \frac{\cos^2\theta'}{a^2} + \frac{\sin^2\theta'}{b^2} = \frac{1}{r'^2};$$

hence (3) becomes $$\frac{x^2}{r^2} + \frac{y^2}{r'^2} = 1,$$

which coincides with the equation in the text, observing that $r = a'$ and $r' = b'$.

306. Cor. 1. Hence, as before, the portions cut off by the tangent from the conjugate axes are

$$\frac{a'^2}{h} \text{ and } \frac{b'^2}{k}.$$

307. Cor. 2. Hence, also, the equation of the line joining the points of contact of the two tangents, drawn from any point (hk), is

$$\frac{hx}{a'^2} + \frac{ky}{b'^2} = 1.$$

308. Cor. 3. Hence we may shew, exactly as in the case of the parabola (see Arts. 236, 237), that the line joining the points of contact of two tangents, drawn from any point of a line parallel to CD, always crosses CP at a given point. Also, that, if two tangents be drawn from any point of a diameter produced, the line joining the points of contact is an ordinate to that diameter.

Various Problems connected with the Ellipse.

309. An ellipse being traced upon paper, to determine its centre.

Draw any two parallel chords PP', QQ' (fig. 94), bisect them in M and N, and draw DD' through M and N; then DD' is a diameter, and therefore its middle point (C) is the centre required.

310. An ellipse and its centre being given, to find its axes.

With the given centre C (fig. 95) as centre, describe a circle cutting the ellipse in P and P', join PP', draw ACA' and BCB' perpendicular and parallel to PP'; then AA' and BB' are the axes required.

311. To find the locus of the middle points of all chords passing through a given point (hk).

If (xy) be the middle point of the chord which makes an angle θ with the axis of x, we have

$$\frac{x \cos\theta}{a^2} + \frac{y \sin\theta}{b^2} = 0 \ldots\ldots\ldots\ldots (1),$$

and if (xy) be a point on the line drawn through (hk) making an angle θ with the axis of x, we have

$$(x-h)\sin\theta - (y-k)\cos\theta = 0 \;\ldots\ldots\ldots (2).$$

Eliminating θ between (1) and (2), we find

$$\frac{x(x-h)}{a^2} + \frac{y(y-k)}{b^2} = 1,$$

or

$$\frac{1}{a^2}\left(x-\frac{h}{2}\right)^2 + \frac{1}{b^2}\left(y-\frac{k}{2}\right)^2 = \frac{h^2}{4a^2} + \frac{k^2}{4b^2},$$

which is the equation of the locus required. It represents an ellipse whose centre is $\left(\frac{h}{2}, \frac{k}{2}\right)$, having its axes parallel to those of the given ellipse, and equal to

$$\frac{a}{2}\sqrt{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}, \quad \frac{b}{2}\sqrt{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}.$$

312. If $\theta = PSS'$, $\theta' = PS'S$, to shew that

$$\tan\frac{\theta}{2}\,.\,\tan\frac{\theta'}{2} = \frac{1-e}{1+e}.$$

We have
$$\cos\theta = \frac{SM}{SP} = \frac{ae+x}{a+ex};$$

$$\therefore \quad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{a+ex-ae-x}{a+ex+ae+x}$$

$$= \frac{a-x}{a+x}\,\frac{1-e}{1+e}.$$

Similarly
$$\cos\theta' = \frac{ae-x}{a-ex},$$

$$\text{and } \therefore \quad \tan^2\frac{\theta'}{2} = \frac{a+x}{a-x}\,\frac{1-e}{1+e}.$$

Hence
$$\tan\frac{\theta}{2}\tan\frac{\theta'}{2} = \frac{1-e}{1+e}.$$

313. If θ and θ' be the angles which two conjugate semi-diameters r and r' make with the axis major, to shew that

$$\frac{\sin(\theta'+\theta)}{\sin(\theta'-\theta)} = \frac{r^2 - r'^2}{a^2 - b^2}.$$

We have $\dfrac{rr'\sin(\theta'+\theta)}{rr'\sin(\theta'-\theta)} = \dfrac{xy' + x'y}{xy' - x'y}$ (see Art. 300)

$$= \frac{ab(\cos\phi\sin\phi' + \cos\phi'\sin\phi)}{ab(\cos\phi\sin\phi' - \cos\phi'\sin\phi)}$$

$$= \frac{\sin(\phi'+\phi)}{\sin(\phi'-\phi)}$$

$$= \cos 2\phi, \quad \text{since } \phi' = \phi + \frac{\pi}{2}.$$

Also $\quad r^2 = a^2\cos^2\phi + b^2\sin^2\phi, \quad r'^2 = a^2\sin^2\phi + b^2\cos^2\phi;$

therefore $\quad r^2 - r'^2 = (a^2 - b^2)\cos 2\phi.$

Hence $\quad \dfrac{\sin(\theta'+\theta)}{\sin(\theta'-\theta)} = \dfrac{r^2 - r'^2}{a^2 - b^2}.$

314. The circle described on $S'P$ as diameter touches the circle described on AA' as diameter.

Let (hk) be P; then the equation of the circle described on $S'P$ as diameter, is

$$\left(x - \frac{ae+h}{2}\right)^2 + \left(y - \frac{k}{2}\right)^2 = \left(\frac{h-ae}{2}\right)^2 + \left(\frac{k}{2}\right)^2,$$

or $$x^2 - (h+ae)x + y^2 - ky + aeh = 0 \quad \ldots\ldots\ldots (1).$$

Also the equation of the circle described on AA' as diameter is $$x^2 + y^2 = a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

At the points of intersection of (1) and (2) we have, subtracting (1) from (2), $\quad (h+ae)x + ky = a^2 + aeh \quad \ldots\ldots\ldots (3),$ which is therefore the equation of the line drawn through the points of intersection.

Now, if p be the perpendicular from C upon (3), we have

$$p^2 = \frac{(a^2+aeh)^2}{(h+ae)^2 + k^2} = \frac{a^2(SP)^2}{(SP)^2} = a^2.$$

Hence (3) is a tangent of (2), which cannot be unless the points of intersection of (1) and (2) coincide. Therefore (1) and (2) touch each other.

315. If $S'P = r$, $PS'A = \theta$; to find r and θ in terms of the eccentric angle (ϕ) of the point P.

If x be the abscissa of P, we have

$$r = a - ex = a(1 - e\cos\phi) \quad \dots\dots\dots \quad (1),$$

$$\cos\theta = \frac{x - ae}{r} = \frac{\cos\phi - e}{1 - e\cos\phi} \quad \dots\dots\dots \quad (2),$$

which are the values of r and θ required. These values are of considerable use in Astronomy; (2) being generally expressed in the form

$$\tan\frac{\theta}{2} = \sqrt{\left(\frac{1+e}{1-e}\right)} \tan\frac{\phi}{2}.$$

316. If PQR (fig. 96) be drawn so that $PQ = b$, $PR = a$, and if the rectangle CU be completed; then PU is the normal at P.

For, ψ being the angle which PU makes with CA, we have

$$\tan\psi = \frac{PM + QU}{CM - CQ};$$

which, if $PQA = \phi$, becomes, since $PQ = b$, $PR = a$,

$$\tan\psi = \frac{b\sin\phi + (a - b)\sin\phi}{a\cos\phi - (a - b)\cos\phi} = \frac{a}{b}\tan\phi.$$

Now, if ψ' be the angle which the normal at P makes with CA, we have

$$\tan\psi' = \frac{a^2y}{b^2x} = \frac{a}{b}\tan\phi.$$

Hence $\psi' = \psi$. Q. E. D.

317. If the axes of an ellipse be in the proportion of $1 : \sqrt{2}$, *any* parabola described on the axis minor as axis, and having its vertex at the centre, will cut the ellipse at right angles.

Let the equations of the ellipse and parabola be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad y^2 = 4mx,$$

and let θ and ϕ be the angles which tangents, drawn to the two curves at their point of intersection, make with the axis of x. Then we have

$$\tan\theta = -\frac{b^2x}{a^2y}, \quad \tan\phi = \frac{2m}{y},$$

$$\text{and} \therefore \tan\theta\tan\phi = -\frac{2mb^2x}{a^2y^2}$$

$$= -\frac{b^2}{2a^2}, \quad \text{since } y^2 = 4mx.$$

Hence, if $b : a :: \sqrt{2} : 1$, we have

$$\tan\theta\tan\phi = -1,$$

and therefore the two tangents are at right angles, or, in other words, the curves cut each other at right angles.

318. To find the locus of the middle point of a chord of the curve $Ax^2 + By^2 = C$, the length of the chord being given.

Proceeding just as in Art. 262, we find

$$r^2(A\cos^2\theta + B\sin^2\theta) + 2(Ah\cos\theta + Bk\sin\theta)r + Ah^2 + Bk^2 - C = 0;$$

and, if we suppose (hk) to be the middle point of the chord $2r$, we have

$$Ah\cos\theta + Bk\sin\theta = 0 \;\ldots\ldots\ldots\ldots (1),$$

and therefore

$$r^2(A\cos^2\theta + B\sin^2\theta) + Ah^2 + Bk^2 - C = 0 \ldots (2).$$

(1) gives $\tan\theta = -\dfrac{Ah}{Bk}$, and then (2) becomes

$$r^2\,\frac{AB^2k^2 + BA^2h^2}{A^2h^2 + B^2k^2} + Ah^2 + Bk^2 - C = 0.$$

Now if we suppose r to be invariable (equal to c suppose), and if we put x and y in place of h and k, this equation becomes

$$(Ax^2 + By^2 - C)(A^2x^2 + B^2y^2 - c^2AB) = c^2ABC \ldots (3),$$

which is the equation required.

319. If we suppose $C = 0$, $A = -a^2$ and $B = 1$, the given equation becomes $(y - ax)(y + ax) = 0$, and (3) becomes

$$(y^2 - a^2x^2)(a^4x^2 + y^2 - a^2c^2) = 0.$$

Hence if lines of a given length ($2c$) be drawn between two lines ($y = ax$) and ($y = -ax$), the locus of their middle points is

$$y^2 - a^2x^2 = 0 \quad \text{and} \quad a^4x^2 + y^2 = a^2c^2,$$

the former equation representing the two lines themselves, and the latter an ellipse whose axes are $\frac{c}{a}$ and ac. The former equation represents the locus of the middle points of those chords which have *both* their extremities on *one* of the lines; and the latter of those which have one extremity on one line and the other on the other.

320. CX (fig. 97) is a fixed line, C a fixed point upon it, CQ, QR two equal lines of a constant length, and P a fixed point on QR: to find the locus of P.

Let $CM (= x)$, $MP (= y)$ be the co-ordinates of P, let $QCR = \theta$, $CQ + QP = a$, and $PR = b$; then we have

$$x = CQ \cos\theta + QP \cos\theta = a\cos\theta, \text{ and } y = b\sin\theta.$$

Hence
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The locus required is therefore an ellipse, whose axes are $CQ + QP$ and PR.

Elliptic compasses might be made upon the principle suggested by this problem.

CHAPTER X.

OF THE HYPERBOLA. PROPERTIES OF THE HYPERBOLA CORRESPONDING TO THOSE OF THE ELLIPSE. ASYMPTOTES. PROBLEMS.

321. The equation of the hyperbola referred to its centre (C) as origin is, by Chap. VI.,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$2a$ being the possible axis (AA'). b does not now represent the portion the curve cuts off from the axis of y, since that axis does not meet the curve.

The polar equation referred to the focus (S) is

$$r = \frac{l}{1 - e \cos \theta},$$

l being the semi-latus-rectum, and e the eccentricity.

We have also, by Art. 176,

$$SC = -ae, \quad EC = -\frac{a}{e}, \quad b^2 = a^2(e^2 - 1), \quad l = -a(1 - e^2) = \frac{b^2}{a}.$$

The curve is perfectly symmetrical with respect to the coordinate axes, so that there is another focus S' and another directrix $E'K'$ (fig. 59) corresponding to S and EK.

We shall now, by means of these equations and formulæ, investigate the various remarkable properties of the hyperbola, many of which correspond so exactly to those of the ellipse, that it will be sufficient merely to state them, and refer to what has been already done in the case of the ellipse in proof of them.

Properties of the Hyperbola not connected with Tangents or Diameters.

PROP. LXXXV.

322. To find the distances of any point of a hyperbola from the two foci.

Let (xy) be any point (P) of the hyperbola; then, since ae and 0 are the co-ordinates of S, we have

$$SP^2 = (x - ae)^2 + y^2;$$

which, since $y^2 = b^2\left(\dfrac{x^2}{a^2} - 1\right) = (e^2 - 1)(x^2 - a^2)$, becomes

$$SP^2 = e^2x^2 - 2aex + a^2 = (a - ex)^2,$$

therefore $$SP = \pm (a - ex);$$

SP here means the absolute distance of P from S; hence, since e is > 1, and $x > a$, and therefore $(a - ex)$ a negative quantity, we must reject the upper sign. We have therefore

$$SP = ex - a \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In like manner we find, since $-ae$ and 0 are the co-ordinates of S',

$$S'P^2 = (x + ae)^2 + y^2,$$
$$= e^2x^2 + 2aex + a^2,$$

and therefore $$S'P = ex + a \ldots\ldots\ldots\ldots\ldots (2);*$$

(1) and (2) are the expressions for the required distances.

323. COR. By subtracting (1) from (2) we find immediately

$$S'P - SP = 2a.$$

---

* (1) and (2) may be deduced geometrically, as follows.

In fig. 98, drawing QPQ' parallel to AA', we have

$$SP = eQP = e(CM - CE)$$
$$= ex - a, \qquad \text{since} \quad CE = \frac{a}{e}.$$

Also $$S'P = eQ'P = e(E'C + CM)$$
$$= ex + a.$$

We may also prove geometrically, that $S'P - SP = 2a$, as follows,

$$S'P - SP = e(Q'P - QP) = e(E'C + EC) = 2a.$$

Hence, in the hyperbola, the *difference* of the distances of any point from S and S' is always equal to the axis major.*

We may obtain the values of SP and $S'P$ very easily from Art. 204, as in the case of the ellipse. See Art. 258.

PROP. LXXXVI.

324. If PSP' be any chord drawn through S, the semi-latus-rectum is a harmonic mean between SP and SP'.

This may be proved exactly as in Art. 207.

PROP. LXXXVII.

325. Two given lines being drawn through the vertex A, to find the equation of the line joining the other two points in which they intersect the hyperbola.

Let the equations of the two lines be

$$y = \alpha(x - a), \qquad y = \alpha'(x - a).$$

Then, as in Art. 260, the equation required is

$$\frac{b^2}{a^2}(x + a) - (\alpha + \alpha')\, y + \alpha\alpha'(x - a) = 0.$$

We may deduce from this equation the same conclusion as in Art. 260.

PROP. LXXXVIII.

326. To determine the length (r) of a line drawn, at an angle θ to the axis of x, from any point (hk) to the hyperbola.

As in Art. 262, we find

$$Ur^2 + 2Vr + W = 0 \; \ldots\ldots\ldots\ldots \; (1),$$

where $U = \dfrac{\cos^2\theta}{a^2} - \dfrac{\sin^2\theta}{b^2}$, $V = \dfrac{h\cos\theta}{a^2} - \dfrac{k\sin\theta}{b^2}$,

$$W = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1.$$

---

* Conversely. To find the locus of a point P (fig. 99), the difference of whose distances from two fixed points S and S' is invariable.

Proceeding as in the Note to Art. 255, we have $r' - r = 2a$, and therefore

$$r'^2 = r^2 + 4a^2 + 4ar,$$

$$\text{also} \quad r'^2 = r^2 + 4c^2 + 4cr\cos\theta, \qquad (PSA = \theta),$$

and therefore, putting $c = ae$, we find

$$r = \frac{a(e^2 - 1)}{1 - e\cos\theta};$$

which, since c is evidently $> a$, and therefore $e > 1$, represents a hyperbola.

Also $$PQ + PQ' = -2\,\frac{V}{U} \ldots\ldots\ldots\ldots \text{(2)},$$

$$PQ \,.\, PQ' = \frac{W}{U} \ldots\ldots\ldots\ldots \text{(3)}.$$

Also the condition that (hk) shall be the middle point of the chord, which makes an angle θ with the axis of x, is

$$\frac{h\cos\theta}{a^2} - \frac{k\sin\theta}{b^2} = 0 \ldots\ldots\ldots\ldots \text{(4)}.$$

PROP. LXXXIX.

327. If QQ', RR' be two chords of a hyperbola, and P their point of intersection, the ratio $PQ\,.\,PQ' : PR\,.\,PR'$ is not altered by moving each chord parallel to itself, and so shifting the position of P in any manner.

The proof of this is exactly the same as in the case of the ellipse. See Art. 265.

PROP. XC.

328. If QQ', RR', Q_1Q_2 be chords of the hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$, and R_1R_2 a chord of the hyperbola $\left(-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$;* Q_1Q_2, R_1R_2 being respectively parallel to QQ', RR', P being the point of intersection of QQ', RR', and P_1 that of Q_1Q_2, R_1R_2: then
$$\frac{PQ\,.\,PQ'}{PR\,.\,PR'} = -\frac{P_1Q_1\,.\,P_1Q_2}{P_1R_1\,.\,P_1R_2}.$$

For, as in Art. 265, we have
$$\frac{PQ\,.\,PQ'}{PR\;PR'} = \frac{b^2\cos^2\phi - a^2\sin^2\phi}{b^2\cos^2\theta - a^2\sin^2\theta};$$

and, in exactly the same way, we find
$$\frac{P_1Q_2\,.\,P_1Q_2}{P_1R_1\,.\,P_1R_2} = \frac{-b^2\cos^2\phi + a^2\sin^2\phi}{b^2\cos^2\theta - a^2\sin^2\theta}.$$

Hence $$\frac{PQ\,.\,PQ'}{PR\,.\,PR'} = -\frac{P_1Q_1\,.\,P_1Q_2}{P_1R_1\,.\,P_1R_2}. \quad \text{Q. E. D.}$$

The latter hyperbola here considered is generally called the *Conjugate* of the former hyperbola, for a reason we shall presently explain. By Art. 180, the *conjugate hyperbola* has the axis of y for its possible axis, as is represented in fig. 100, where B_1, B_2 are the vertices of the conjugate hyperbola, and $B_1B_2 = 2b$.

* See Art. 180, Equation 3.

Prop. XCI.

329. To find the equation of the hyperbola referred to the centre as pole.

As in Art. 266, we find the required equation to be

$$r = \frac{ab}{\sqrt{(b^2 \cos^2 \theta - a^2 \sin^2 \theta)}}.$$

Of the Eccentric Angle in the Hyperbola. (See Art. 268.)

Prop. XCII.

330. The co-ordinates x and y of any point of a hyperbola may be put in the form

$$x = a \cos \phi, \qquad y = b\sqrt{(-1)} \sin \phi.$$

As in the case of the ellipse, we may assume $\frac{x}{a} = \cos \phi$; only, in the present case, ϕ is an imaginary quantity, since x is always greater than a*. Making this assumption we find, from the equation of the hyperbola,

$$-\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \sin^2 \phi, \text{ and therefore } y = b\sqrt{(-1)} \sin \phi.$$

Hence, if we assume $x = a \cos \phi$, we find $y = b\sqrt{(-1)} \sin \phi$. Q. E. D.

331. Or, we may proceed somewhat differently, thus,

Let ψ be the logarithm of $\frac{x}{a} + \frac{y}{b}$, which gives

$$\frac{x}{a} + \frac{y}{b} = e^{\psi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

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* We must regard $\cos \phi$ here as an abbreviation for the expression $\frac{1}{2}\{e^{\phi\sqrt{(-1)}} + e^{-\phi\sqrt{(-1)}}\}$. Indeed we ought to consider the *general* definitions of the functions $\sin \phi$ and $\cos \phi$ to be

$$\cos \phi = \tfrac{1}{2}\{e^{\phi\sqrt{(-1)}} + e^{-\phi\sqrt{(-1)}}\}, \quad \sin \phi = \frac{1}{2\sqrt{(-1)}}\{e^{\phi\sqrt{(-1)}} - e^{-\phi\sqrt{(-1)}}\}:$$

from which definitions all the properties of sines and cosines may be very readily deduced; as, for instance, the properties

$$\sin^2 \phi + \cos^2 \phi = 1, \qquad \sin(\phi + \phi') = \sin \phi \cos \phi' + \cos \phi \sin \phi'.$$

It is easy to see from these definitions that, when ϕ is real, $\cos \phi$ and $\sin \phi$ are real, and both less than unity: but, when ϕ is imaginary, $\cos \phi$ is real, $\sin \phi$ is imaginary, and $\cos \phi$ is greater than unity.

and then, since, by the equation of the hyperbola,

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=1, \text{ we find}$$

$$\frac{x}{a}-\frac{y}{b}=e^{-\psi} \dots\dots\dots\dots\dots\dots\dots (2),$$

(1) + (2), and (1) − (2) give

$$x=\tfrac{1}{2}a\,(e^{\psi}+e^{-\psi}) \dots. (3), \qquad y=\tfrac{1}{2}b\,(e^{\psi}-e^{-\psi}) \dots. (4).$$

To express these formulæ more concisely, let us put $\phi\sqrt{(-1)}$ for ψ, and then, by the exponential values of $\sin\phi$ and $\cos\phi$, we have

$$x=a\cos\phi \dots. (5), \qquad y=b\sqrt{(-1)}\sin\phi \dots. (6).$$

Of course we are as much at liberty to use the expressions (5) and (6) in place of (3) and (4), as we are to employ the common exponential formulæ for the sine and cosine of an angle.

Properties of the Hyperbola connected with Tangents.

PROP. XCIII.

332. To find the angle which the tangent at any point (hk) of a hyperbola makes with the axis of x.

As in Art. 274, we find

$$\tan\theta=\frac{b^2h}{a^2k},$$

which determines the angle required.*

Hence the equation of the tangent at (hk) is

$$\frac{hx}{a^2}-\frac{ky}{b^2}=1.$$

Hence we have $\quad CT=\dfrac{a^2}{h}, \quad CT'=-\dfrac{b^2}{k}.$

333. COR. Hence if $(\alpha x+\beta y=\gamma)$ be the equation of a tangent to the hyperbola, we have the condition

$$\gamma^2=a^2\alpha^2-b^2\beta^2.$$

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* The Notes to Arts. 275, 277 apply equally well to the case of the hyperbola, putting $-b^2$ for b^2.

Prop. XCIV.

334. To find the normal at any point (hk) of the hyperbola, and the portions it cuts off from the axes.

As in Art. 279, we find the equation required to be

$$a^2k(x-h)+b^2h(y-k)=0,$$

or $$\frac{a^2}{h}x+\frac{b^2}{k}y=a^2+b^2.$$

Hence $$CG=\frac{a^2+b^2}{a^2}h=e^2h,$$

$$CG'=\frac{a^2+b^2}{b^2}k=\frac{a^2}{b^2}e^2k.$$

335. Cor. Hence the normal at P makes equal angles with SP and $S'P$.

For $SG=CG-CS=e^2h-ae=e\,.\,SP$, by Art. 321;

and $S'G=CG+CS=e^2h+ae=e\,.\,S'P$;

therefore $SG:S'G=SP:S'P$;

which shews that the angle SPS' is bisected by PG, and therefore that the normal makes equal angles with SP and $S'P$.

336. Cor. Hence the tangent at P bisects the angle SPS'. This result may be proved geometrically from the property, $S'P-SP=2a$, as in the case of the parabola. See Note to Art. 217.

Prop. XCV.

337. To find the locus of the intersection of a tangent of a hyperbola with a perpendicular let fall upon it from either focus.

As in Art. 282, we may shew that the locus required is the circle $$x^2+y^2=a^2.*$$

338. Cor. Hence we may derive a geometrical method of drawing a tangent from any given point to a hyperbola exactly the same as that explained in Art. 283.

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* This may be shewn geometrically as follows. (See Note, Art. 283.)

Let SY (fig. 101) be the perpendicular from S upon the tangent (YP); produce SY to meet $S'P$ in Q. Then we have $QPY=SPY$, and therefore $PQ=PS$; which gives $S'Q=S'P-SP=2a$: and since $SC=S'C$, $SY=QY$, we have $CY=\frac{1}{2}S'Q=a$. Hence, &c. &c.

PROP. XCVI.

339. To determine analytically the tangents of a hyperbola which pass through any given point.

As in Art. 284, we may determine the required tangents from the equation

$$\alpha^2 + \frac{2hk}{a^2 - h^2}\,\alpha + \frac{-b^2 - k^2}{a^2 - h^2} = 0.$$

340. COR. Hence the locus of the point of intersection (hk) of two tangents at right angles to each other is represented by

$$h^2 + k^2 = a^2 - b^2,$$

which is a circle; unless b^2 be greater than a^2, in which case the locus is impossible; *i.e.* two tangents cannot be drawn at right angles to each other when b^2 is greater than a^2.

PROP. XCVII.

341. If p and p' be the perpendiculars from S and S' upon any tangent, to shew that $pp' = b^2$.

As in Art. 286, we have

$$pp' = \frac{\gamma^2 - a^2(\alpha^2 + b^2)}{\alpha^2 + \beta^2} \qquad \gamma^2 = a^2\alpha^2 - b^2\beta^2,$$

and therefore $$pp' = -b^2;$$

the sign of pp' is negative because the two perpendiculars lie on opposite sides of the tangent. We shall suppose, however, that p and p' represent the absolute lengths of the perpendiculars without regard to sign, and then we have $pp' = b^2$.

342. COR. As in Art. 287, we have $\frac{p}{p'} = \frac{r}{r'}$; and, multiplying this by $pp' = b^2$, we find

$$p^2 = b^2\frac{r}{r'} = b^2\frac{r}{r + 2a}.$$

343. Arts. 288–291 hold with so little variation in the case of the hyperbola, that it is unnecessary to repeat them here.

Properties of the Hyperbola connected with Diameters.

PROP. XCVIII.

344. To determine the diameter of a given system of parallel chords in a hyperbola.

As in Art. 292, the equation of the diameter is

$$\frac{x \cos\theta}{a^2} - \frac{y \sin\theta}{b^2} = 0 \;\ldots\ldots\ldots\ldots\ (1).$$

Art. 293 is equally true for the hyperbola; and the same may be said of Art. 294.

As in Art. 295, we may shew that

$$\tan\theta \,.\, \tan\theta' = \frac{b^2}{a^2} \;\ldots\ldots\ldots\ldots\ldots\ (2).$$

Hence if θ be $< 90°$, θ' is also $< 90°$; *i.e.* the diameter and its chord make an acute angle with each other.

Art. 296 is equally true in the case of the hyperbola, and conjugate diameters are defined exactly as in the case of the ellipse.

345. As in Art. 298, if (xy) and $(x'y')$ be any points on two conjugate diameters respectively, we have

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

Prop. XCIX.

346. A diameter and its conjugate cannot both meet the hyperbola.

If possible let them meet the hyperbola in the points (xy) and $(x'y')$; then from (3) we have

$$\frac{x^2}{a^2}\frac{x'^2}{a^2} = \frac{y^2}{b^2}\frac{y'^2}{b^2}$$

$$= \left(\frac{x^2}{a^2} - 1\right)\left(\frac{x'^2}{a^2} - 1\right),$$

since (xy) and $(x'y')$ are points on the hyperbola:

therefore $$x^2 + x'^2 = a^2;$$

which is impossible, since x^2 and x'^2 are always each greater than a^2. Hence only one of two conjugate diameters meets the hyperbola.

347. It is easy to see that if a diameter do not meet the hyperbola itself, it will meet the conjugate hyperbola.

348. When a diameter does not meet the hyperbola, we shall define the *extremities* of that diameter to be the points where it meets the *conjugate* hyperbola.

349. Cor. Hence, if (xy) be an extremity (P) of a diameter which meets the hyperbola, and if $(x'y')$ be an extremity (D) of its conjugate diameter, which, as we have shewn, meets the conjugate hyperbola, but not the hyperbola itself; we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots\ldots (1), \quad -\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \ldots\ldots (2),$$

and, by Art. 344,
$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0 \ldots\ldots\ldots\ldots (3).$$

Prop. C.

350. To determine the relation between the eccentric angles belonging to the extremities P and D of two conjugate diameters of a hyperbola.

If in Art. 348 we assume $x = a \cos \phi$, $y' = b \cos \phi'$*, (1) and (2) give $y = b\sqrt{(-1)} \sin \phi$, $x' = a\sqrt{(-1)} \sin \phi'$: which values put in (3), give $\sin \phi \cos \phi' - \cos \phi \sin \phi' = 0$, or $\sin (\phi - \phi') = 0$; hence $\phi' = \phi$; which is the required relation.†

351. Cor. Hence
$$x' = a\sqrt{(-1)} \sin \phi = \frac{a}{b}\, y,$$
$$y' = \quad b \cos \phi \quad = \frac{b}{a}\, x,$$
which formulæ determine the co-ordinates of D in terms of those of P.

Prop. CI.

352. The difference of the squares of CP and CD, and the rectangle completed upon CP and CD, are invariable.

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* We assume $y' = b \cos \phi'$, and not $x' = a \cos \phi'$, because the y' in (2) corresponds to the x in (1), since the axis of y is the possible axis of the conjugate hyperbola.

† This may be shewn, without introducing imaginary quantities, as follows.

Assume $2x = a(e^{\psi} + e^{-\psi})$, and $\therefore 2y = b(e^{\psi} - e^{-\psi})$,

also $2y' = b(e^{\psi'} + e^{-\psi'})$, and $\therefore 2x' = a(e^{\psi'} - e^{-\psi'})$;

then, substituting these values in (3), Art. 348, we have

$$(e^{\psi} + e^{-\psi})(e^{\psi'} - e^{-\psi'}) - (e^{\psi} - e^{-\psi})(e^{\psi'} + e^{-\psi'}) = 0,$$

or $$e^{\psi - \psi'} = e^{\psi' - \psi},$$

which gives $\psi - \psi' = \psi' - \psi$, or $\psi' = \psi$.

Let $CP = r$, $CD = r'$, $PCX = \theta$, $DCX = \theta'$, (fig. 100); then

$$r^2 = x^2 + y^2 = \quad a^2 \cos^2 \phi - b^2 \sin^2 \phi,$$
$$r'^2 = x'^2 + y'^2 = -a^2 \sin^2 \phi + b^2 \cos^2 \phi;$$

therefore $$r^2 - r'^2 = a^2 - b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (1).$$

Again, if A be the area of the parallelogram completed upon CP and CD, we have

$$\begin{aligned} A = rr' \sin(\theta' - \theta) &= rr' (\sin \theta' \cos \theta - \cos \theta' \sin \theta) \\ &= xy' - x'y \\ &= ab (\cos^2 \phi + \sin^2 \phi). \end{aligned}$$

Hence $$A = ab \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2).$$

(1) shews that the difference of the squares of CP and CD, and (2) that the area of the parallelogram completed upon CP and CD, are invariable.

353. Cor. As in the case of the ellipse, we have

$$r^2 = e^2x^2 - b^2, \qquad r'^2 = e^2x^2 - a^2.$$

354. Cor. Also $\quad SP \,.\, S'P = CD^2.$

Prop. CII.

355. To find the equation of the hyperbola referred to two conjugate diameters, CP and CD, as co-ordinate axes (fig. 102).

Draw any chord QMQ' parallel to DCD'; assume $CM = x$, $MQ = y$, $CP = a'$, $CD = b'$, Then, by Art. 327, we have

$$\frac{MP \,.\, MP'}{MQ \,.\, MQ'} = -\frac{CP \,.\, CP'}{CD \,.\, CD'}.$$

But $CP = a'$, $MP = x - a'$, $MP' = x + a'$, $CD = b'$, $MQ = y$, $CP' = -CP$, $CD' = -CD$, and $MQ' = -MQ$; therefore

$$\frac{x^2 - a'^2}{a'^2} = \frac{y^2}{b'^2},$$

or $$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1,$$

which is the equation required.*

* As in the case of the ellipse, by transforming the axes, we may prove that the equation of the hyperbola referred to two conjugate axes is

$$\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right)x^2 + \left(\frac{\cos^2\theta'}{a^2} - \frac{\sin^2\theta'}{b^2}\right)y^2 = 1;$$

Hence it appears that the equation of the hyperbola referred to any two conjugate diameters is of exactly the same form as that referred to CA and CB.

356. Arts. 305–308 are true in the case of the hyperbola, if we put $-b^2$ for b^2.

Properties of the Hyperbola connected with Asymptotes.

Prop. CIII.

357. To determine the asymptotes of a hyperbola.

A tangent is said to become an *asymptote* when the distance of the point of contact from the origin becomes infinite, provided at the same time, the tangent itself remains at a finite distance from the origin.

If $r\cos\theta$ and $r\sin\theta$ be the co-ordinates of the point of contact, we have, by the equation of the tangent (Art. 331), and by the equation of the hyperbola,

$$\frac{x\cos\theta}{a^2} - \frac{y\sin\theta}{b^2} = \frac{1}{r} \ldots\ldots\ldots\ldots (1),$$

$$\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \frac{1}{r^2} \ldots\ldots\ldots\ldots (2),$$

(2) shews that, when r becomes infinite,

$$\frac{\cos\theta}{a} = \pm\frac{\sin\theta}{b};$$

and hence, when r becomes infinite, the equation (1) gives us

$$\frac{x}{a} \pm \frac{y}{b} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It appears therefore that, when the point of contact moves to an infinite distance from the origin, the equation of the tangent assumes the form (3), which represents two right lines at a

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which, since $\dfrac{\cos^2\theta}{a^2} - \dfrac{\sin^2\theta}{b^2} = \dfrac{1}{CP^2}$, and $-\dfrac{\cos^2\theta'}{a^2} + \dfrac{\sin^2\theta'}{b^2} = \dfrac{1}{CD^2}$,

becomes $\dfrac{x^2}{a'^2} - \dfrac{y^2}{b'^2} = 1.$

finite distance from the origin. Hence the hyperbola has two asymptotes represented by the equations

$$\frac{x}{a} + \frac{y}{b} = 0, \qquad \frac{x}{a} - \frac{y}{b} = 0.$$

358. Cor. 1. If we draw HAH' (fig. 103) through A at right angles to AA', and take $AH = AH' = b$, it is evident that the equations of the lines CH, CH' are

$$\frac{x}{a} - \frac{y}{b} = 0, \qquad \frac{x}{a} + \frac{y}{b} = 0.$$

Hence these lines are the two asymptotes of the hyperbola.

Cor. 2. When $a = b$ the asymptotes are right angles to each other. In this case the hyperbola is said to be *equilateral.*

Prop. CIV.

359. The ellipse and the parabola have no asymptotes.

The equation to the ellipse gives

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{r^2},$$

and therefore r can never become infinite, since we should then have

$$\tan^2\theta = -\frac{b^2}{a^2}.$$

Proceeding in the case of the parabola as above, we have

$$y \sin\theta = 2m\left(\frac{x}{r} + \cos\theta\right) \dots\dots\dots\dots (1),$$

$$\sin^2\theta = \frac{4m\cos\theta}{r} \dots\dots\dots\dots\dots\dots (2).$$

If r become infinite, (2) gives $\sin\theta = 0$, and then (1) becomes $0 = 2m$, which is absurd; *i.e.* (1) cannot be satisfied by any finite values of x and y when $r = \infty$.

Hence, neither the ellipse nor the parabola have asymptotes.

Prop. CV.

360. To find the length of a line (r) drawn, at an angle θ to the axis of x, from any point (hk) to either of the asymptotes.

The equation of the two asymptotes considered as one locus is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \dots\dots\dots\dots\dots\dots (1).$$

Hence, proceeding exactly as in Art. 262, we find

$$Ur^2 + 2Vr + W = 0 \;\ldots\ldots\ldots\ldots\; (2),$$

when $U = \dfrac{\cos^2\theta}{a^2} - \dfrac{\sin^2\theta}{b^2}$, $V = \dfrac{h\cos\theta}{a^2} - \dfrac{k\sin\theta}{b^2}$,

$$W = \frac{h^2}{a^2} - \frac{k^2}{b^2},$$

which equation determines the required length, giving of course two values of it.

361. Cor. 1. Hence the condition that (hk) may be the middle point of a line intercepted between the two asymptotes, making an angle θ with the axis of x, is

$$\frac{h\cos\theta}{a^2} - \frac{k\sin\theta}{b^2} = 0 \;\ldots\ldots\ldots\ldots\; (3).$$

Now this is also the condition (see Art. 325) that (hk) may be the middle point of a chord of the hyperbola making the same angle θ with the axis of x. Hence, if $RQPQ'R'$ (fig. 105) be any line meeting the two asymptotes at R, R', and the hyperbola at Q, Q', and if P be the middle point of QQ'; P will also be the middle point of RR'.

362. Cor. 2. Hence RQ is always equal to $R'Q'$, whatever line $RQQ'R'$ may be.

363. Cor. 3. (3) gives, $\tan\theta = \dfrac{b^2h}{a^2k}$; hence, by Art. 331, if (hk) be a point on the hyperbola, the portion of the tangent at (hk) which is intercepted between the asymptotes is bisected at (hk). This may be easily shewn from Cor. 2, by supposing $RQPQ'R'$ to move parallel to itself, till Q and Q' (and therefore P) coincide.

Prop. CVI.

364. To find the equation of the hyperbola referred to its asymptotes as co-ordinate axes.

Let $\tan\alpha = \dfrac{b}{a}$, then the two asymptotes make angles α and $-\alpha$ with the axis of x. Hence, by Art. 110, if we put

$(x + y) \cos \alpha$ for x,* and $(-x + y) \sin \alpha$ for y, in the equation of the hyperbola, viz.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots (1),$$

we obtain the required equation. Now, if we assume $m^2 = a^2 + b$, we have $\cos \alpha = \frac{a}{m}$, $\sin \alpha = \frac{b}{m}$; hence, making the substitutions in (1), it becomes

$$(x + y)^2 - (x - y)^2 = m^2,$$

or

$$xy = \frac{m^2}{4} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which is therefore the equation of the hyperbola referred to its asymptotes as co-ordinate axes.

Prop. CVII.

365. To find the equation of the tangent at any point when the asymptotes are the co-ordinate axes.

Proceeding as in the note to Art. 275, we have

$$y - k = \frac{k' - k}{h' - h}(x - h) \ldots\ldots\ldots\ldots (3),$$

and $hk = \frac{m^2}{4}$, $h'k' = \frac{m^2}{4}$, by (2), last Art.,

and therefore

$$k' - k = \frac{m^2}{4}\left(\frac{1}{h'} - \frac{1}{h}\right) = -\frac{m^2}{4h'h}(h' - h).$$

Hence (3) becomes $y - k = -\frac{m^2}{4hh'}(x - h)$;

and therefore, reasoning as in the note referred to, the equation of the tangent at (hk) is

$$(y - k) = -\frac{m^2}{4h^2}(x - h)$$

$$= -\frac{k}{h}(x - h) \text{ since } m^2 = 4hk,$$

or

$$\frac{x}{2h} + \frac{y}{2k} = 1,$$

which is the equation required.

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* This is easily seen from fig. 104, where CH, CH' are the asymptotes, P any point of the hyperbola, $CM' (= x')$, $M'P (= y')$ the co-ordinates of P referred to CH, CH', and $CM (= x)$, $MP (= y)$ the old co-ordinates.

366. Cor. 1. Hence the portions, CT, CT' (fig. 106), which the tangent cuts off from the axes, are $2h$ and $2k$; which evidently shews that TT' is bisected at the point of contact P.

367. Cor. 2. Hence the area of the triangle CTT' is invariable; for it is equal to $\frac{1}{2}\,CT.CT'\sin TCT'$, which $= 2hk \sin 2a$ $= \frac{4hkab}{m^2}\left(\text{since } \sin a = \frac{b}{m}, \cos a = \frac{a}{m}\right) = ab\left(\text{since } hk = \frac{m^2}{4}\right)$.

Prop. CVIII.

368. The product of the two perpendiculars let fall from any point of the hyperbola upon the asymptotes is invariable.

The equations of the asymptotes being

$$\frac{x}{a} + \frac{y}{b} = 0, \quad \frac{x}{a} - \frac{y}{b} = 0,$$

the perpendiculars upon them from any point (hk) are

$$p = \frac{1}{c}\left(\frac{h}{a} + \frac{k}{b}\right), \text{ and } p' = \frac{1}{c}\left(\frac{h}{a} - \frac{k}{b}\right), \text{ where } c^2 = \frac{1}{a^2} + \frac{1}{b^2}.$$

and therefore $$pp' = \frac{1}{c^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)$$

$$= \frac{1}{c^2}, \text{ if } (hk) \text{ be a point on the hyperbola.}$$

Hence $$pp' = \frac{a^2b^2}{a^2+b^2}. \quad \text{Q. E. D.}$$

Various Problems respecting the Hyperbola.

369. A line AB (fig. 107) is drawn meeting the axes of co-ordinates OX, OY (which are supposed to be oblique) at A and B; if AOB be always a given area, to find the locus of the middle point (P) of AB.

If x, y be the co-ordinates of P, we have

$$OA = 2x, \quad OB = 2y, \quad \text{and } \therefore \text{ area } AOB = 4xy \sin \omega,$$

where $\omega = \angle XOY$. Hence, we have

$$xy = \frac{c^2}{4 \sin \omega},$$

where c^2 is the given area.

The locus required is therefore a hyperbola whose asymptotes are OX, OY.

370. To find the foci of the rectangular hyperbola, $xy = m^2$, from the definition given in Art. 205.

Let (hk) be either focus, then we have

$$(x-h)^2 + (y-k)^2 = (Ax + By - C)^2,$$

and this equation must be identical with the equation $xy = m^2$; we have therefore

$$A^2 = 1, \quad B^2 = 1, \quad h = AC, \quad k = BC, \quad h^2 + k^2 - C^2 = 2ABm^2.$$

From which it is easy to see that

$$h = m\sqrt{2}, \quad k = m\sqrt{2}; \quad \text{or } h = -m\sqrt{2}, \quad k = -m\sqrt{2}.$$

371. OX, OY (fig. 107) are two given right lines, and C a given point; to find the locus of the middle point P of the right line AB, which always passes through C.

Let the co-ordinates of C be h and k, and those of P, x and y; then $OA = 2x$, $OB = 2y$, and therefore, by Art. 57, we have

$$\frac{h}{2x} + \frac{k}{2y} = 1,$$

$$\text{or} \quad 2xy - hy - kx = 0.$$

This equation may be put in the form

$$2\left(x - \frac{h}{2}\right)\left(y - \frac{k}{2}\right) = \frac{hk}{2},$$

which, transferring the origin to the point $\left(\frac{h}{2}\,\frac{k}{2}\right)$, becomes

$$xy = \frac{hk}{4}.$$

If, therefore, we join O and C, and bisect OC in M, the locus required is a hyperbola, whose centre is M, and whose asymptotes are parallel to OX and OY.

372. The base of a triangle and the difference of the base angles are given, to find the locus of the vertex.

Take the base as axis of x, and its middle point as origin, and let the equations of the sides be

$$y = m(x-a) \ldots\ldots\ldots(1), \quad y = m'(x+a) \ldots\ldots\ldots(2).$$

Let β be the given difference of the base angles, then we have

$$\frac{(-m) - m'}{1 + (-m)m'} = \tan\beta \ldots\ldots\ldots\ldots\ldots(3);$$

and, if we eliminate m and m' from these three equations, the result will be the equation of the locus required. Hence, substituting the values of m and m', given by (1) and (2), in (3), we find the equation of the locus to be

$$\frac{y}{x-a} + \frac{y}{x+a} + \tan\beta\left(1 - \frac{y^2}{x^2-a^2}\right),$$

or $$x^2 + 2\cot\beta \,.\, xy - y^2 = a^2.$$

Now, when the axes are turned through an angle θ, let this equation become $$Ax^2 + Cy^2 = a^2 \;\ldots\ldots\ldots\ldots\ldots (4);$$
then (see Art. 190) we have

$$A\cos^2\theta + C\sin^2\theta = 1,$$
$$(A - C)\cos\theta\sin\theta = \cot\beta,$$
$$A\sin^2\theta + C\cos^2\theta = -1;$$

and from these equations we find

$$A + C = 0, \quad (A - C)\cos 2\theta = 2, \quad \tan 2\theta = \cot\beta.$$

Hence $$2\theta = \frac{\pi}{2} - \beta, \quad A = \frac{1}{\sin\beta}, \quad C = -\frac{1}{\sin\beta},$$

and therefore (4) becomes

$$x^2 - y^2 = a^2\sin\beta.$$

Hence the locus required is an equilateral hyperbola whose possible axis makes an angle $\frac{\pi}{4} - \frac{\beta}{2}$ with the base of the triangle.

373. This problem may be solved more readily as follows.

Let the axis of y be inclined at an angle ω to the base, the axis of x coinciding with it as before, and let θ, ϕ be the angles which the sides make with the axis of x. Then the equations of the sides are

$$y = \frac{\sin\theta}{\sin(\omega-\theta)}(x-a) \ldots (1), \quad y = \frac{\sin\phi}{\sin(\omega-\phi)}(x+a) \ldots (2);$$

and we have $$(\pi - \theta) - \phi = \beta.$$

Hence, if we assume the arbitrary angle ω to be equal to $\pi - \beta$, and therefore $\phi = \omega - \theta$, (2) becomes

$$y = \frac{\sin(\omega-\theta)}{\sin\theta}(x+a) \ldots\ldots\ldots\ldots (3),$$

and (3) × (1) gives $\quad x^2 - y^2 = a^2.$
Hence the locus is an hyperbola, and the axes of co-ordinates are a pair of conjugate diameters.

374. All the ellipses that may be described on the asymptotes of a given hyperbola as conjugate axes, and touching the hyperbola, have the same area.

Let the equation of the hyperbola, the asymptotes being co-ordinate axes, be $\quad xy = m^2,$
and the equation of the ellipse

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

Supposing x and y to have the same values in these two equations, we have

$$\frac{x^2}{a'^2} - \frac{xy}{m^2} + \frac{y^2}{b'^2} = 0;$$

and, if the ellipse touch the hyperbola, this latter equation ought to give two equal values of $\frac{y}{x}$. Therefore we find

$$\frac{1}{a'^2} \cdot \frac{1}{b'^2} = \left(\frac{1}{2m^2}\right)^2,$$

or

$$a'b' = 2m^2.$$

Now, if A be the area of the ellipse, we have *$A = \pi ab$, and therefore, since $ab = a'b' \sin \omega$,

$$A = \pi a'b' \sin \omega = 2m^2 \sin \omega,$$

ω being the angle of ordination. Hence A is invariable.

375. To find the locus of the point of intersection of two tangents of a hyperbola which are parallel to two conjugate diameters.

If $\tan^{-1}\alpha$ be the angle which a tangent, drawn from the point (xy) to the hyperbola, makes with the axis of x, we have, by Art. 338,

$$\alpha^2 - \frac{2xy}{x^2 - a^2}\,\alpha + \frac{y^2 + b^2}{x^2 - a^2} = 0.$$

Let α and α' be the two values of α given by this equation, then

$$\alpha\alpha' = \frac{y^2 + b^2}{x^2 - a^2}.$$

* That $A = \pi ab$, will be proved in Art. 391.

But if the two corresponding tangents be parallel to a pair of conjugate diameters, we have

$$aa' = \frac{b^2}{a^2}.$$

Hence
$$\frac{y^2 + b^2}{x^2 - a^2} = \frac{b^2}{a^2},$$

or
$$\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1;$$

which is the equation of a hyperbola, whose axes are $a\sqrt{2}$ and $b\sqrt{2}$.

COR. The corresponding locus in the case of an ellipse is

$$\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1.$$

376. If two tangents be drawn from any point (Q) of either of the asymptotes of a hyperbola, to shew that the tangent of the angle they make with each other is

$$\frac{ab}{r^2 - a^2 + b^2};$$

r being the distance of Q from the centre.

CHAPTER XI.

MISCELLANEOUS PROPOSITIONS.

Prop. CIX.

377. To obtain the relations between the co-ordinates of the extremities of two conjugate diameters in an ellipse or hyperbola without using the eccentric angle.

Let (xy) be P, and $(x'y')$ D; then, as in Art. 298, we have

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

Let us assume that $\frac{x'}{a} = u\frac{y}{b}$, u being some unknown quantity, then (1) gives $\frac{y'}{b} = -u\frac{x}{a}$. Now, by the equation of the ellipse, we have

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1,$$

$$\therefore u^2\left(\frac{y^2}{b^2} + \frac{x^2}{a^2}\right) = 1, \quad \text{and } \therefore u^2 = 1, \quad \text{or } u = \pm 1.$$

Hence we have

$$\frac{x'}{a} \pm \frac{y}{b} = 0, \quad \frac{x}{a} \mp \frac{y'}{b} = 0,$$

which are the relations required.

The double signs refer to the two extremities of each diameter: it is usual to suppose P and D to be the extremities to which the upper signs belong: on this supposition we have

$$\frac{x'}{a} + \frac{y}{b} = 0 \ldots (2), \qquad \frac{x}{a} - \frac{y'}{b} = 0 \ldots (3).$$

In the case of the hyperbola we have

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0.$$

Assume $\frac{x'}{a} = u\,\frac{y}{b}$, and $\therefore \frac{y'}{b} = u\,\frac{x}{a}$; then, since $(x'y')$ is a point of the conjugate hyperbola, we have

$$-\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1,$$

$$\therefore\ u^2\left(-\frac{y^2}{b^2} + \frac{x^2}{a^2}\right) = 1;$$

which, since (xy) is a point on the hyperbola itself, gives $u^2 = 1$, and $\therefore u = \pm 1$, as before. Hence we have

$$\frac{x'}{a} \pm \frac{y}{b} = 0, \quad \frac{x}{a} \pm \frac{y'}{b} = 0;$$

or, omitting the upper signs,

$$\frac{x'}{a} - \frac{y}{b} = 0\ldots.(4), \quad \frac{x}{a} - \frac{y'}{b} = 0\ldots.(5).$$

PROP. CX.

378. From the preceding results to deduce the properties of conjugate diameters.

From (2) and (3), we have

$$x'^2 + y'^2 = a^2\,\frac{y^2}{b^2} + b^2\,\frac{x^2}{a^2}$$

$$= a^2\left(1 - \frac{x^2}{a^2}\right) + b^2\left(1 - \frac{y^2}{b^2}\right),$$

$$\therefore\ x'^2 + y'^2 = a^2 + b^2 - (x^2 + y^2),$$

or

$$CD^2 + CP^2 = a^2 + b^2.$$

Again, from (2) and (3), we have

$$xy' - x'y = \frac{bx^2}{a} + \frac{ay^2}{b}$$

$$= ab\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right),$$

$$\therefore\ xy' - x'y = ab.$$

In this result, putting $r\cos\theta$, $r\sin\theta$ for x, y; and $r'\cos\theta'$, $r'\sin\theta'$ for x', y'; we have

$$rr'\sin(\theta'-\theta)=ab,$$

i.e. the area of the parallelogram completed upon CP and CD is invariable.

In the case of the hyperbola we may shew, in exactly the same way, that

$$x'^2+y'^2=a^2\frac{y^2}{b^2}+b^2\frac{x^2}{a^2}$$

$$=a^2\left(\frac{x^2}{a^2}-1\right)+b^2\left(\frac{y^2}{b^2}+1\right),$$

$$\therefore\quad x'^2+y'^2=x^2+y^2-(a^2-b^2),$$

or

$$CP^2-CD^2=a^2-b^2.$$

Again

$$xy'-x'y=\frac{bx^2}{a}-\frac{ay^2}{b}$$

$$=ab\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),$$

$$\therefore\quad xy'-x'y=ab;$$

and hence

$$rr'\sin(\theta'-\theta)=ab.$$

Prop. CXI.

379. A conic section being referred to any axes of co-ordinates originating at the centre; to determine the radius of the circle, having its centre at the origin, which touches the conic section.

The general equation of a conic section, referred to the centre as origin, is

$$Ax^2+2Bxy+Cy^2=1 \ldots\ldots\ldots\ldots (1).$$

Let the equation of the circle, whose radius we wish to determine, be

$$x^2+y^2=r^2\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$(1)-\frac{1}{r^2}(2)$ gives

$$\left(A-\frac{1}{r^2}\right)x^2+2Bxy+\left(C-\frac{1}{r^2}\right)y^2=0\ldots\ldots(3).$$

In general this equation gives two different values of $\frac{y}{x}$; but these values will become equal when the circle touches the

conic section, therefore the first member of the equation will be a perfect square, and we shall have

$$\left\{A - \frac{1}{r^2}\right\}\left\{C - \frac{1}{r^2}\right\} = B^2,$$

or $$\frac{1}{r^4} - (A + C)\frac{1}{r^2} + AC - B^2 = 0 \;\ldots\ldots\ldots\; (4),$$

which equation, therefore, determines the radius of the circle required.

380. Cor. 1. When a circle, concentric with a conic section, touches it, the points of contact must evidently be at the extremities of either the major or minor axis (or the extremities of the possible axis in the case of a hyperbola): hence the values of r given by (4) are the semiaxes, a and b, of the conic section; we have, therefore,

$$\frac{1}{a^2} + \frac{1}{b^2} = A + C, \qquad \frac{1}{a^2b^2} = AC - B^2.$$

Cor. 2. Hence, if AC be $> B^2$ and $A + C > 0$, a^2 and b^2 are both positive, and therefore the curve is an ellipse; but if AC be $< B^2$ either a^2 or b^2 is negative, and the curve is therefore a hyperbola.

Prop. CXII.

381. To determine the position of the axes of the conic section in the preceding proposition.

Supposing the first member of (3) to be a perfect square, we have

$$\left\{A - \frac{1}{r^2}\right\}x + By = 0, \qquad \left\{C - \frac{1}{r^2}\right\}y + Bx = 0;*$$

and, multiplying the first of these equations by y, and the second by x, and subtracting, we find

$$(A - C)\,xy + B\,(y^2 - x^2) = 0:$$

and, putting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\tan 2\theta = \frac{2B}{A - C},$$

* For if $Lx^2 + 2Mxy + Ny^2$ be a perfect square, its square root is either

$$\sqrt{L}\,.\,x + \frac{M}{\sqrt{L}}\,.\,y, \quad \text{or } \sqrt{N}\,.\,y + \frac{M}{\sqrt{N}}\,.\,x.$$

which determines two values of θ, whose difference is $90°$, and these values are the angles which the two axes of the conic section make with the axis of x.

PROP. CXIII.

382. To determine the magnitude and position of the axes of the conic section represented by $Ax^2 + 2Bxy + Cy^2 = 1$, when the co-ordinates are oblique, the angle of ordination being ω.

We may proceed exactly as in the preceding proposition, only instead of the equation (2) we now have

$$x^2 + 2xy \cos \omega + y^2 = r^2;$$

and therefore, instead of (3), we have

$$\left(A - \frac{1}{r^2}\right) x^2 + 2\left(B - \frac{\cos \omega}{r^2}\right) xy + \left(C - \frac{1}{r^2}\right) y^2 = 0 \ldots (5),$$

$$\therefore \left(A - \frac{1}{r^2}\right)\left(C - \frac{1}{r^2}\right) = \left(B - \frac{\cos \omega}{r^2}\right)^2,$$

or
$$\frac{1}{r^4} - \frac{A + C - 2B \cos \omega}{\sin^2 \omega} \cdot \frac{1}{r^2} + \frac{AC - B^2}{\sin^2 \omega} = 0.$$

Hence $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{A + C - 2B \cos \omega}{\sin^2 \omega}$, $\dfrac{1}{a^2 b^2} = \dfrac{AC - B^2}{\sin^2 \omega}$.

Again, to determine the position of the axes, we obtain from (5), as in Prop. CXII.,

$$Ax + By - (x + y \cos \omega) \frac{1}{r^2} = 0,$$

$$Cy + Bx - (y + x \cos \omega) \frac{1}{r^2} = 0,$$

and $\therefore$ $(Ax + By)(y + x \cos \omega) - (Cy + Bx)(x + y \cos \omega) = 0$,

or $(A \cos \omega - B) x^2 + (A - C) xy + (B - C \cos \omega) y^2 = 0$.

This equation gives two values of $\dfrac{y}{x}$, $\dfrac{p}{q}$, $\dfrac{p'}{q'}$ suppose, and we have, therefore, either

$$px - qy = 0, \quad \text{or } p'x - q'y = 0,$$

which are evidently the equations of the two axes.

383. COR. If we suppose $A = \dfrac{1}{a'^2}$, $C = \dfrac{1}{b'^2}$, $B = 0$, the equation of the conic section becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

and (5) becomes

$$\left(\frac{r^2}{a'^2} - 1\right)x^2 - 2\cos\omega\,.\,xy + \left(\frac{r^2}{b'^2} - 1\right)y^2 = 0,$$

and $\therefore$
$$\left(\frac{r^2}{a'^2} - 1\right)\left(\frac{r^2}{b'^2} - 1\right) = \cos^2\omega,$$

or
$$r^4 - (a'^2 + b'^2)\,r^2 + a'^2b'^2\sin^2\omega = 0.$$

Hence $\quad a'^2 + b'^2 = a^2 + b^2 \quad$ and $\quad a'b'\sin\omega = ab,$

which are the two properties of conjugate diameters previously obtained by different methods.

Prop. CXIV.

384. A tangent is drawn from a given point (xy) to an ellipse, to find the angle (ψ) which the portion of it between (xy) and the point of contact (hk) subtends at either focus.

Let r and r' be the distances of (xy) and (hk) from the focus S'; then we evidently have

$$\cos\psi = \frac{x - ae}{r}\,.\,\frac{h - ae}{r'} + \frac{y}{r}\,.\,\frac{k}{r'} \,\ldots\ldots\ldots(1),$$

and
$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \,\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Substituting in (1) the value of ky obtained from (2), we have

$$rr'\cos\psi = (x - ae)(h - ae) + b^2 - \frac{b^2}{a^2}hx$$
$$= x(e^2h - ae) - aeh + a^2, \quad \text{since } b^2 = a^2(1 - e^2),$$
$$= (a - ex)(a - eh):$$

and hence, since $r' = a - eh$, we have

$$\cos\psi = \frac{a - ex}{r},$$

which determines the required angle.

385. Cor. 1. If Q (fig. 108) be the point (xy), P the point (hk), and QRM an ordinate; then $S'Q = r$ and $S'R = a - ex$; hence

$$\cos PS'Q = \frac{S'R}{S'Q};$$

and hence, if we draw QT perpendicular to $S'P$, we have

$$S'T = S'R.$$

386. Cor. 2. If QP' be the other tangent drawn from Q, we have $\cos QS'P' = \dfrac{a - ex}{r}$. Hence, if from any point Q two tangents, QP, QP', be drawn to an ellipse, the lines QP and QP' subtend equal angles at either focus.

387. Cor. 3. The angle ψ being given to find the locus of Q.

We have $$r \cos \psi = a - ex,$$

or, transferring the origin to S',

$$r \cos \psi = a - e(x + ae) = a(1 - e^2) - ex;$$

which, introducing polar co-ordinates, and putting $\dfrac{a(1 - e^2)}{\cos \psi} = l'$, $\dfrac{e}{\cos \psi} = e'$ becomes $$r = \frac{l'}{1 + e' \cos \theta};$$

which is the equation of the locus required; it is therefore a conic section having its focus at S', and its eccentricity and latus rectum equal to $\dfrac{e}{\cos \psi}$ and $\dfrac{l}{\cos \psi}$ respectively.

Prop. CXV.

388. Having given the magnitude of position of two conjugate diameters of an ellipse, to find the magnitude and position of the axes.

Let r, r' be the two given conjugate semi-diameters, ω the angle they make with each other, and θ, θ' the angles they make respectively with the axis major. Then we have

$$r^2 + r'^2 = a^2 + b^2 \ldots\ldots\ldots\ldots\ldots (1),$$

$$2rr' \sin \omega = 2ab \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

therefore, adding and subtracting, and putting, for brevity,

$$r^2 + 2rr' \sin \omega + r'^2 = u^2, \qquad r^2 - 2rr' \sin \omega + r'^2 = v^2,$$

we have $$a \pm b = u \pm v,$$

$$\text{and } \therefore\ 2a = u + v, \qquad 2b = u - v;$$

which determine a and b in terms of r, r' and ω.

To find θ we have, since $\theta' = \theta + \omega$,

$$\tan \theta \tan (\theta + \omega) = -\frac{b^2}{a^2},$$

which determines θ.

PROP. CXVI.

389. To find the magnitude and position of the two equal conjugate diameters.

Using the notation in the preceding proposition, and putting $r' = r$, we have, by (1) and (2),

$$r^2 = \frac{a^2 + b^2}{2};$$

$$\sin \omega = \frac{2ab}{a^2 + b^2},$$

which determine r and ω. To determine θ we have

$$\frac{r^2 \cos^2 \theta}{a^2} + \frac{r^2 \sin^2 \theta}{b^2} = 1 = \cos^2 \theta + \sin^2 \theta;$$

$$\therefore \quad \frac{2r^2 - 2a^2}{a^2} \cos^2 \theta + \frac{2r^2 - 2b^2}{b^2} \sin^2 \theta = 0,$$

or $$\frac{b^2 - a^2}{a^2} \cos^2 \theta + \frac{a^2 - b^2}{b^2} \sin^2 \theta = 0, \text{ since } 2r^2 = a^2 + b^2.$$

Hence $\tan \theta = \pm \dfrac{b}{a}$, which determines the position of the two diameters.

PROP. CXVII.

390. To find the greatest and least values of ω.

We have $\sin \omega = \dfrac{ab}{rr'}$, and therefore $\sin \omega$ is a maximum or minimum according as rr', or $(rr')^2$ is a maximum or minimum.

Now $$r^2 r'^2 = r^2 (a^2 + b^2 - r^2) = \left(\frac{a^2 + b^2}{2}\right)^2 - \left(r^2 - \frac{a^2 + b^2}{2}\right)^2,$$

which is evidently greatest when $r^2 = \dfrac{a^2 + b^2}{2}$, and therefore $r'^2 = \dfrac{a^2 + b^2}{2}$ also, and least when r^2 has its greatest value a^2. Hence $\sin \omega$ is least when $r = r'$, and greatest when $r = a$; and therefore, since we suppose ω to be greater than $\dfrac{\pi}{2}$, ω is greatest when $r = r'$, and least when $r = a$.

Hence the minimum value of ω is $\dfrac{\pi}{2}$, and the maximum value is given by the equation

$$\sin \omega = \frac{2ab}{a^2 + b^2}.$$

Prop. CXVIII.

391. To find the area of an ellipse.

If we inscribe in the elliptic quadrant ABC, (fig. 109) the rectilineal figure $pqp'q'p''q''$......; pq, $p'q'$, &c. being parallel to CA, and qp', $q'p''$, &c. perpendicular to CA; it is clear that we may make the area of the rectilineal figure differ from that of the elliptic quadrant as little as we please, by taking each of the sides pq, $p'q'$, &c. sufficiently small. Let (xy), $(x'y')$, $(x''y'')$, &c. be the points q, q', q'', &c., X the area of the rectilineal figure, and A that of the ellipse: then we have

$$X = y(x - 0) + y'(x' - x) + y''(x'' - x') + \&c.;$$

and if X' be the corresponding rectilineal figure inscribed in the circular quadrant ACB', we have, by Art. 261,

$$X' = \frac{a}{b}y(x - 0) + \frac{a}{b}y'(x' - x) + \frac{a}{b}y''(x'' - x') + \&c.;$$

$$\therefore\ X = \frac{b}{a}X'.$$

Hence $\frac{b}{a}X'$ may be made to differ from A as little as we please: but X' may be made to differ from the area of $AB'C$, *i.e.* $\frac{\pi a^2}{4}$, as little as we please; therefore the difference between A and $\frac{b}{a}\cdot\frac{\pi a^2}{4}$ must be less than any small quantity we choose to specify: which cannot be, unless $A = \frac{b}{a}\frac{\pi a^2}{4}$. Therefore $A = \frac{\pi ab}{4}$, and hence the area of the whole ellipse is πab.

392. Cor. 1. Hence, if we produce the ordinate MP (fig. 110) to meet the circle described on AA' as diameter in Q, it is clear that

$$\text{area } A'MP = \frac{b}{a}\text{ area } A'MQ.$$

Also $$\text{area } CMP = \frac{MP}{MQ}\text{ area } CMQ = \frac{b}{a}\text{ area } CMQ,$$

$$\therefore\ \text{area } CPA' = \frac{b}{a}\text{ area } CQA'$$

$$= \frac{b}{a}\,\frac{a^2\phi}{2}, \quad \text{if } \angle QCA' = \phi;$$

hence the expression for the area CPA' is

$$\tfrac{1}{2}\,\phi ab.$$

393. Cor. 2. If we put $b\sqrt{(-1)}$ for b, and ψ for $\phi\sqrt{(-1)}$, we have

$$\text{area } CPA' = \tfrac{1}{2}\,\psi ab\,;$$

which is the expression for the hyperbolic area CPA' (fig. 111), where

$$\psi = \log\left(\frac{x}{a} + \frac{y}{b}\right), \quad \text{(see Art. 330).}$$

Prop. CXIX.

394. To find the area of the portion AMP of a parabola (fig. 112), MP being any ordinate.

Draw PN parallel to AM, let p, p' be any two points of the arc AP whose co-ordinates are xy, $x'y'$, and draw $q'pm$, $p'qm'$ parallel to AN, and qpn, $p'q'n'$ parallel to AM. Then, if we denote the areas $mpp'm'$ and $npp'n'$ by X and Y, we have

$$X = y\,(x' - x) + \text{area } pp'q, \quad Y = x\,(y' - y) + \text{area } pp'q'.$$

Hence it is clear that $\dfrac{X}{Y}$ may be made to differ from $\dfrac{y\,(x' - x)}{x\,(y' - y)}$ as little as we please by bringing p' up to p. Now, since $y^2 = 4mx$, and $y'^2 = 4mx'$, we have

$$\frac{y\,(x' - x)}{x\,(y' - y)} = \frac{y}{4mx} \cdot \frac{y'^2 - y^2}{y' - y} = \frac{y' + y}{y}\,;$$

and $\dfrac{y' + y}{y}$ may be made to differ from 2 as little as we please by bringing p' up to p. It follows therefore that $\dfrac{X}{Y}$ may be made to differ from 2 as little as we please, by bringing p' up to p.

Now we may divide the area AMP into a series of portions such as X, and the area ANP into another series of portions such as Y, and by what has been proved, we may make the ratio of each of the former series to the corresponding one of the latter differ from 2 as little as we please. Hence the ratio of the area AMP to the area ANP must differ from 2 by a quantity less than any small quantity we choose to specify: which cannot be unless area $AMP = 2$ area ANP, or, what is the same thing, area $AMP = \tfrac{2}{3}$ area $AMPN$.

Hence it appears, that the area of the parabolic segment APM is two thirds of the circumscribing rectangle $AMPN$.

Prop. CXX.

395. Two chords, drawn from the extremities of the major axis to any point of an ellipse, are always parallel to a pair of conjugate diameters.

Let (hk) be the point to which the chords are drawn, then the equations of the chords are

$$\frac{x-a}{h-a}=\frac{y}{k}, \quad \frac{x+a}{h+a}=\frac{y}{k}.$$

Hence, if θ and θ' be the angles which these two lines make with the axis of x, we have

$$\tan\theta=\frac{k}{h-a}, \quad \tan\theta'=\frac{k}{h+a}.$$

$$\text{and } \therefore \quad \tan\theta\tan\theta'=-\frac{k^2}{a^2-h^2},$$

$$=-\frac{b^2}{a^2}, \text{ since } \frac{h^2}{a^2}+\frac{k^2}{b^2}=1,$$

which shews that the two chords are parallel to a pair of conjugate diameters.

396. Cor. In the same way we may shew that, if two chords be drawn from the extremities of any diameter to any point of an ellipse, they are parallel to a pair of conjugate diameters. Only instead of $\tan\theta$ and $\tan\theta'$ we must write $\frac{\sin\theta}{\sin(\omega-\theta)}$ and $\frac{\sin\theta'}{\sin(\omega-\theta')}$, and a' and b' for a and b. We also assume that if θ and θ' be the angles which any pair of conjugate axes make with the axis of x, when the ellipse is referred to a pair of conjugate axes, then

$$\frac{\sin\theta}{\sin(\omega-\theta)}\cdot\frac{\sin\theta'}{\sin(\omega-\theta')}=-\frac{b'^2}{a'^2}.$$

This may be proved, as in the case of the ellipse referred to the axis major and minor, by putting $h+r\frac{\sin(\omega-\theta)}{\sin\omega}$ and $h+r\frac{\sin\theta}{\sin\omega}$ for x and y in the equation $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$; and then putting the coefficient of r, in the result, equal to zero.

CHAPTER XII.

OF THE GENERAL EQUATION OF THE SECOND DEGREE BETWEEN x AND y.

397. In the following propositions we shall consider the equation of the second degree in its most general form

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

We have already (in Chap. VII.) investigated the nature of the locus represented by this equation, and proved that it represents in general an ellipse, parabola, hyperbola, two right lines, one right line, or an isolated point. We have also given a method of determining the nature and position of the locus when the coefficients A, B, C, &c. are given; and shewed that this method is simplified by first reducing the equation to the form

$$Ax^2 + 2Bxy + Cy^2 + F' = 0,$$

i. e. by transferring the origin to the centre, which may always be done, except when $B^2 - AC = 0$. We shall now pursue this subject more into detail.

Prop. CXXI.

398. To determine the length (r) of a right line drawn, at an angle θ to the axis of x, from a point (hk) to the locus represented by the equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

As in Arts. 68, 128, &c. put $h + r\cos\theta$ and $k + r\sin\theta$ for x and y in the equation of the locus

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \;\ldots\ldots\; (1),$$

and it becomes $\qquad Ur^2 + 2Vr + W = 0 \;\ldots\ldots\ldots\ldots\ldots\; (2),$

where

$$\left.\begin{aligned} U &= A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta \\ V &= (Ah + Bk + D)\cos\theta + (Ck + Bh + E)\sin\theta \\ W &= Ah^2 + 2Bhk + Ck^2 + 2Dh + 2Ek + F \end{aligned}\right\}\ldots(3);$$

(2) determines the required distance r, and in general gives two values of it.

Using the notation in Arts. 129, 211, &c. we have

$$\left.\begin{aligned} PQ + PQ' &= -\frac{V}{U} \\ PQ.PQ' &= \frac{W}{U} \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(4).$$

399. Cor. From the equation (4) we may, just as in Arts. 132, 133, 135, &c., draw the following conclusions:

(1) That the point (hk) bisects the chord drawn through it at an angle θ to the axis of x, when $V = 0$, *i. e.* when

$$(Ah + Bk + D)\cos\theta + (Bk + Ch + E)\sin\theta = 0\ldots(5).$$

(2) That the equation of the diameter of the system of chords inclined at an angle θ to the axis of x, is

$$(Ax + By + D)\cos\theta + (Cy + Bx + E)\sin\theta = 0,$$

or $(A\cos\theta + B\sin\theta)x + (C\sin\theta + B\cos\theta)y + D\cos\theta + E\sin\theta = 0\ldots(6)$.

(3) That the angle (θ) which the tangent, at the point (hk) of the locus, makes with the axis of x, is given by the equation $V = 0$, or

$$\tan\theta = -\frac{Ah + Bk + D}{Ck + Bh + E}\ldots\ldots\ldots\ldots(7).$$

(4) That hence the equation of the tangent at the point (hk) is*

$$(Ah + Bk + D)(x - h) + (Ck + Bh + E)(y - k) = 0,$$

* We may find the equation of the tangent in the following manner also, as in the Notes to Articles 214, 275.

Let (hk) and $(h'k')$ be two points of the locus; then

$$Ah^2 + 2Bhk + Ck^2 + 2Dh + 2Ek + F = 0\ldots\ldots\ldots\ldots(1).$$

$$Ah'^2 + 2Bh'k' + Ck'^2 + 2Dh' + 2Ek' + F' = 0\ldots\ldots\ldots\ldots(2).$$

Subtracting (1) from (2), and observing that

$$h'k' - hk = h'k' - h'k + h'k - hk = h'(k' - k) + k(h' - h),$$

we have, dividing by $h' - h$,

$$A(h' + h) + 2Bk + 2D + \{C(k' + k) + 2Bh' + 2E\}\frac{k' - k}{h' - h} = 0.$$

which, in virtue of the equation (1), since (hk) is a point of the locus, becomes

$$(Ah + Bk + D)\,x + (Ck + Bh + E)\,y + Dh + Ek + F = 0 \ldots (8).$$

(5) That hence the equation of the line joining the points of contact of the two tangents drawn from any point (hk) to the locus, is

$$(Ax + By + D)\,h + (Cy + Bx + E)\,k + Dx + Ey + F = 0,$$

or $(Ah + Bk + D)\,x + (Ck + Bh + E)\,y + Dh + Ek + F = 0 \ldots (9).$

(6) That this last equation is also the equation of the locus of the point of intersection of the pair of tangents drawn at the extremities of any chord passing through the point (hk).

(7) From the second of the equations (4) we may shew, just as in Arts. 213, &c., that, if QQ', RR' be two chords of the locus, and P their point of intersection, the ratio $PQ.PQ' : PR.PR'$ is not altered by moving the chords parallel to themselves in any manner.

Prop. CXXII.

400 To find the centre of the locus represented by the general equation of the second degree.

The centre is that point which bisects all the chords drawn through it; therefore, if (hk) be the centre, the equation (5), Art. 399, must be true for all values of θ. Hence we have

$$Ah + Bk + D = 0 \ldots (1), \qquad Ck + Bh + E = 0 \ldots (2);$$

which two equations determine h and k, and therefore the centre of the locus. If we actually calculate h and k from these equations, we find

$$h = \frac{BE - CD}{AC - B^2}, \qquad k = \frac{BD - AE}{AC - B^2}.$$

Cor. 1. If $AC - B^2 = 0$, h and k are in general infinite, *i.e.* the locus has no centre. We may also see this from the

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Hence the equation of the line joining (hk) and $(h'k')$ is

$$y - k = -\frac{A(h' + h) + 2Bk + 2D}{C(k' + k) + 2Bh' + 2E}(x - h);$$

which, when h' and k' approach h and k, becomes in the limit

$$y - k = -\frac{Ah + Bk + D}{Ck + Bh + E}(x - h),$$

which is the equation of the tangent obtained in the text.

equations (1) and (2), Art. 400; for, when $B^2 = AC$, they become $\left(\text{putting } C = \frac{B^2}{A}\right)$

$$Ah + Bk + D = 0, \quad Ah + Bk + \frac{EA}{B} = 0,$$

which equations are manifestly inconsistent, unless $D = \frac{EA}{B}$, and therefore cannot be satisfied by finite values of h and k. Consequently there can be no centre when $B^2 = AC$, except in the particular case where $D = \frac{EA}{B}$.

401. COR. 2. In the particular case just mentioned, the equations (1) and (2) become identical, and therefore equivalent to only one equation between h and k. Consequently an infinite number of values of h and k will satisfy the conditions (1) and (2),* and therefore the locus will have an infinite number of centres, whose co-ordinates are restricted to satisfy the equation $Ah + Bk + D = 0$; *i.e.* all these centres are situated on the right line represented by $Ax + By + D = 0$.

We may verify this result in the following manner. If $B^2 = AC$, $BD = EA$, and $\therefore BE = CD$, the general equation of the second degree becomes

$$\frac{BD}{E}x^2 + 2Bxy + \frac{BE}{D}y^2 + 2Dx + 2Ey + F = 0,$$

or

$$\frac{B}{DE}(Dx + Ey)^2 + 2(Dx + Ey) + F = 0.$$

If M and N be the two values of $Dx + Ey$ given by this equation, we have

$$(Dx + Ey - M)(Dx + Ey - N) = 0.$$

Hence the general equation represents two parallel right lines. Now it is evident that any point of the right line, drawn parallel to these two lines, and equidistant from them, is a centre of the two lines considered as one locus. Hence there are an infinite number of centres all situated in a right line.

* If $B^2 = AC$, and $BD = AE$, then $BE = CD$ also, and the expressions for h and k, instead of being infinite, assume the form $\frac{0}{0}$; which shews that h and k are indeterminate.

Prop. CXXIII.

402. To find what chords of the locus, represented by the general equation of the second degree, are perpendicular to their diameters.

By equation (6), Art. 399, if θ be the angle which a chord makes with the axis of x, the angle (θ') which its diameter makes with the axis of x, is given by the equation

$$\tan\theta' = -\frac{A\cos\theta + B\sin\theta}{C\sin\theta + B\cos\theta};$$

and if the diameter and chord be at right angles, we have

$$\tan\theta' = -\cot\theta.$$

Hence $(C\sin\theta + B\cos\theta)\cos\theta = (A\cos\theta + B\sin\theta)\sin\theta$,

and $\therefore$ $B(\cos^2\theta - \sin^2\theta) = (A - C)\sin\theta\cos\theta$,

or $$\tan 2\theta = \frac{2B}{A - C};$$

which equation gives two values of θ differing by 90° from each other. It is the same equation as that obtained in Art. 187 for determining the position of the axes of the locus. Hence there are only two diameters, namely the axes, whose chords are at right angles to them.

Prop. CXXIV.

403. To shew that the locus represented by the equation of the second degree is in general an ellipse, parabola, or hyperbola, according as B^2 is less than, equal to, or greater than, AC.

Recurring to the equations (3), (4), Art. 398, we find that, in order to make P the middle point of every chord which passes through P, we must have $PQ = -PQ'$, or $V = 0$, for all values of θ, and therefore

$$Ah + Bk + D = 0, \qquad Ck + Bh + E = 0.$$

Now these equations are inconsistent when $\frac{A}{B} = \frac{B}{C}$, *i.e.* when $B^2 = AC$ $\left(\text{except } \frac{D}{B} = \frac{E}{C} \text{ also}\right)$. Hence, in general, when $B^2 = AC$, there is no point which bisects all the chords drawn through it; and consequently the locus has no centre; *i.e.* it must be a parabola.

But if B^2 be not equal to AC, we may always determine h and k from the two equations just put down; therefore the locus has a centre, and it must be, in general, an ellipse or hyperbola. To distinguish which it is, we have only to suppose that $QQ' = \infty$, which can only be the case in a hyperbola, and therefore that $PQ \,.\, PQ' = \infty$; which gives $U = 0$, or

$$A \cos^2 \theta + 2B \cos \theta \sin \theta + C \sin^2 \theta = 0,$$

and $\therefore$
$$\tan \theta = -\frac{B \pm \sqrt{(B^2 - AC)}}{C}.$$

Now this value of θ is possible only when B^2 is $> AC$; and therefore the locus in an ellipse or hyperbola, according as B^2 is $<$ or $> AC$.*

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* We may obtain this result somewhat differently, as follows:

For each value of x two values of y are given by the quadratic equation

$$Cy^2 + 2(Bx + E)y + Ax^2 + 2Dx + F = 0;$$

and these values are real, provided

$$(Bx + E)^2 \text{ be } > C(Ax^2 + 2Dx + F),$$

or
$$(B^2 - AC) + 2(BE - CD)\frac{1}{x} + (E^2 - CF)\frac{1}{x^2} > 0.$$

Now, by assuming x to be a sufficiently large positive or negative quantity, we may make the last two terms of the first side of this inequality as small as we please, and therefore the first side to have the same sign as $B^2 - AC$. Hence, if B^2 be $< AC$, the inequality is not satisfied by very large positive or negative values of x; but, if B^2 be $> AC$, the equation is satisfied by any very large positive or negative values of x.

If $B^2 = AC$, the condition that y may be possible becomes

$$x\left\{2(BE - CD) + \frac{E^2 - CF}{x}\right\} > 0;$$

and therefore, if $BE - CD$ be positive, y is possible for very large values of x, provided they be positive; and if $BE - CD$ be negative, y is possible for very large values of x, provided they be negative.

Exactly the same reasoning will apply if we solve the general equation for x instead of y.

Hence the locus has no points indefinitely distant from the axes of coordinates when B^2 is less than AC; it has points indefinitely distant from the axes on both sides of each of them when B^2 is greater than AC; and it has points indefinitely distant from the axes, but only on one side of each of them when B^2 is equal to AC. Therefore the form of the locus corresponds with that of an ellipse, parabola, or hyperbola, according as B^2 is less than, equal to, or greater than AC.

Prop. CXXV.

404. To determine under what circumstances the locus represented by the general equation of the second degree is two parallel right lines.

If the locus be two parallel right lines, every point of the right line drawn parallel to, and equidistant from them, is a centre, and therefore the locus has an infinite number of centres. Consequently the equations

$$Ah + Bk + D = 0, \qquad Ck + Bh + E = 0,$$

must be satisfied by an infinite number of different values of h and k; which cannot be unless the two equations be identical. We have, therefore,

$$\frac{A}{B} = \frac{B}{C} = \frac{D}{E},$$

or $\qquad AC = B^2, \quad$ and $BE = CD$ (or $EA = BD$).

If these conditions hold, the general equation becomes

$$\frac{BD}{E} x^2 + 2Bxy + \frac{BE}{D} y^2 + 2Dx + 2Ey + F = 0,$$

or
$$\frac{B}{ED} (Dx + Ey)^2 + 2 (Dx + Ey) + F = 0,$$

or
$$\frac{B}{ED} (Dx + Ey) + 1 \pm \sqrt{\left(1 - \frac{BF}{ED}\right)} = 0.$$

If BF be $< ED$, this equation represents two parallel right lines; but, if BF be $> ED$, it represents no locus.

Hence the conditions necessary in order that the general equation of the second degree may represent two parallel right lines, are
$$\frac{A}{B} = \frac{B}{C} = \frac{D}{E}, \quad \text{and } BF < ED.$$

Prop. CXXVI.

405. To determine the circumstances under which the general equation of the second degree represents two intersecting right lines, or an isolated point.

When the locus is two intersecting right lines, or an isolated point, the centre is manifestly a point of the locus. Now, if we transfer the origin to the centre, the general equation becomes

$$Ax^2 + 2Bxy + Cy^2 + Dh + Ek + F = 0 \ldots\ldots (1),$$

(see Art. 193 *bis*) where h and k are given by the equations

$$Ah + Bk + D = 0 \ldots (2), \qquad Ck + Bh + E = 0 \ldots (3).$$

Hence, since the centre, which is now the origin, is a point of the locus, (1) must be satisfied by the values $x = 0$, $y = 0$; therefore we have $\quad Dh + Ek + F = 0 \ldots\ldots\ldots\ldots\ldots (4),$

and
$$Ax^2 + 2Bxy + Cy^2 = 0 \ldots$$

or
$$x + \frac{B \pm \sqrt{(B^2 - AC)}}{A} y = 0 \ldots\ldots\ldots (5).$$

Also, if we substitute the values of h and k, got from (2) and (3) in (4), we find

$$CD^2 - 2BDE + AE^2 + F(B^2 - AC) = 0 \ldots (6);$$

(6) is therefore the condition necessary in order that the locus may be two intersecting right lines or an isolated point. If B^2 be $> AC$, (5) represents two right lines intersecting at the centre; but if B^2 be $< AC$, (5) is satisfied only by $x = 0$ and $y = 0$, and therefore represents an isolated point, namely the centre. Hence the locus will be two intersecting right lines, or an isolated point, according as B^2 is greater or less than AC.

Prop. CXXVII.

406. To determine whether the locus represented by the general equation of the second degree has asymptotes; and if so, to find their equations.

If we put $r \cos\theta$ and $r \sin\theta$ for h and k, in the general equation, $\quad Ah^2 + 2Bhk + Ck^2 + 2Dh + 2Ek + F = 0,$

and, in the equation of the tangent,

$$(Ah + Bk + D)\,x + (Ck + Bh + E)\,y + Dh + Ek + F = 0,$$

they become

$$A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta + \frac{2}{r}(D\cos\theta + E\sin\theta) + \frac{F}{r^2} = 0,$$

and
$$(Ax + By + D)\cos\theta + (Cy + Bx + E)\sin\theta + \frac{F}{r} = 0.$$

Hence, if r become infinite, *i.e.* if the point of contact be at an infinite distance from the origin, we have

$$A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta = 0,$$

and
$$(Ax + By + D)\cos\theta + (Cy + Bx + E)\sin\theta = 0.$$

The first of these equations gives

$$A \cos\theta + \{B \pm \sqrt{(B^2 - AC)}\} \sin\theta = 0,$$

and, thence, the second becomes

$$(Ax + By + D)\{B \pm \sqrt{(B^2 - AC)}\} - (Cy + Bx + E)A = 0 \ldots (1),$$

which is, in general, the equation of the tangent when the point of contact is at an infinite distance from the origin.

Hence, when B^2 is $> AC$, the locus has two asymptotes, represented by (1); and when B^2 is $< AC$, it has no asymptote. If $B^2 = AC$, (1) becomes

$$0 . x + 0 . y + DB - AE = 0,$$

which shews that the tangent goes off to an infinite distance from the origin, for this equation cannot be satisfied by any finite values of x and y.* Therefore the locus has, in general, no asymptotes when $B^2 = AC$.

407. Cor. It is evident, from the form of the equation (1), that the two asymptotes intersect at the centre; for the co-ordinates of the centre satisfy the equations

$$Ax + By + D = 0, \qquad Cy + Dx + E = 0.$$

Prop. CXXVIII.

408. To obtain the equations of the asymptotes independently of the equation of the tangent.

Assuming the notation in Art. 398, we have

$$Ur^2 + 2Vr + W = 0,$$

or

$$U + 2V . \frac{1}{r} + W . \frac{1}{r^2} = 0.$$

Now if $U = 0$ and $V = 0$, this equation makes each of the two values of $\frac{1}{r}$ equal to zero: hence, if $U = 0$ and $V = 0$, the two points, where the line, drawn from the point (hk) at an angle θ to the axis of x, meets the locus, are at an infinite distance from (hk); under which circumstances this line is evidently an asymptote. Hence the equations $U = 0$ and $V = 0$ make (hk) a point on an asymptote of the locus. If, therefore, we eliminate θ from

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* Except when $DB - AE = 0$, in which case the locus is two parallel right lines.

these two equations, we find a general relation between the co-ordinates, h and k, of any point on an asymptote of the locus. Hence, putting for U and V their values, and substituting x and y for h and k, the asymptotes are determined by eliminating θ from the equations

$$A\cos^2\theta + 2B\sin\theta\cos\theta + C\sin^2\theta = 0,$$
$$(Ax + By + D)\cos\theta + (Cy + Bx + E)\sin\theta = 0,$$

which agrees with the preceding proposition.

Prop. CXXIX.

409. To shew that if the equation of the second degree be put in the form $(\alpha x + \beta y + \gamma)(\alpha' x + \beta' y + \gamma') = \eta$, the asymptotes are represented by the equations, $\alpha x + \beta y + \gamma = 0$, and $\alpha' x + \beta' y + \gamma' = 0$.

Putting $h + r\cos\theta$, and $k + r\sin\theta$, for x and y, the equation of the locus becomes
$$Ur^2 + 2Vr + W = 0,$$
where $U = (\alpha\cos\theta + \beta\sin\theta)(\alpha'\cos\theta + \beta'\sin\theta)$,

$V = (\alpha\cos\theta + \beta\sin\theta)(\alpha' h + \beta' k + \gamma') + (\alpha'\cos\theta + \beta'\sin\theta)(\alpha k + \beta k + \gamma) = 0$.

To obtain the equations of the asymptotes, we have only to put $U = 0$, and $V = 0$, and eliminate θ from these two equations. Now $U = 0$ gives $\alpha\cos\theta + \beta\sin\theta = 0$, or $\alpha'\cos\theta + \beta'\sin\theta = 0$; and therefore $V = 0$ gives $\alpha k + \beta k + \gamma = 0$, or $\alpha' h + \beta' k + \gamma' = 0$. Hence, putting x and y for h and k, the equations of the two asymptotes are

$$\alpha x + \beta y + \gamma = 0, \qquad \alpha' x + \beta' y + \gamma' = 0.$$

Prop. CXXX.

410. To put the general equation of the second degree in the form $(\alpha x + \beta y + \gamma)(\alpha' x + \beta' y + \gamma') = \eta$.

Transferring the origin to the centre, the general equation of the second degree becomes (see Art. 193, *bis*)

$$Ax^2 + 2Bxy + Cy^2 + Dh + Ek + F = 0 \; .\;.\;.\;. \; (1);$$

h and k, the co-ordinates of the centre, being given by the equations
$$Ah + Bk + D = 0, \qquad Ck + Bh + E = 0.$$

Now
$$Ax^2 + 2Bxy + Cy^2 = Ay^2\left\{\frac{x^2}{y^2} + 2\frac{B}{A}\frac{x}{y} + \frac{C}{A}\right\}$$
$$= Ay^2\left(\frac{x}{y} - \lambda\right)\left(\frac{x}{y} - \mu\right),$$

when λ and μ are the roots of $z^2 + 2\frac{B}{A}z + \frac{C}{A} = 0$.

Hence (1) becomes

$$A(x - \lambda y)(x - \mu y) + Dh + Ek + F = 0;$$

Now transferring the origin back to its old position, *i. e.* putting $x - h$ and $y - k$ for x and y, this equation becomes

$$A\{x - \lambda y + h - \lambda k\}\{x - \mu y + h - \mu k\} + Dh + Ek + F = 0,$$

which is the equation (1) put in the required form, h and k being given by the equations

$$Ah + Bk + D = 0, \qquad Ck + Bh + E = 0,$$

and λ, μ being the roots of $Az^2 + 2Bz + C = 0$.

Cor. Hence the equations of the asymptotes are

$$(x - h) - \lambda(y - k) = 0, \qquad (x - h) - \mu(y - k) = 0.$$

Prop. CXXXI.

411. To trace the form of the locus represented by the general equation of the second degree, by solving the equation for y.

The general equation of the second degree may be put in the form

$$y^2 + 2\frac{Bx + E}{C}y + \frac{Ax^2 + 2Dx + F}{C} = 0,$$

which gives

$$y = -\frac{Bx + E}{C} \pm \frac{\sqrt{\{(B^2 - AC)x^2 + 2(BE - DC)x + E^2 - CF}}{C},$$

$$= -\frac{Bx + E}{C} \pm R \text{ suppose.}$$

Let AB (fig. 113) be the right line represented by the equation $Cy + Bx + E = 0$; then, if we take $OM = x$, and draw the ordinate MPQ to meet AB in P, it is evident that

$$MP = -\frac{Bx + E}{C}.$$

Hence, if we take $PQ = R$, and $PQ' = -R$, we have

$$MQ = -\frac{Bx + E}{C} + R, \qquad MQ' = -\frac{Bx + E}{C} - R.$$

It follows therefore that Q and Q' are the points where the ordinate drawn from M meets the locus, and this is generally true for all positions of M. Also, since $PQ = PQ'$, it appears

that the line AB is the diameter of the chords of the locus which are parallel to the axis of y.

To make out how the locus lies with respect to AB, we have only to trace the values of R corresponding to different values of x in the following manner.

1st. Let $B^2 = AC$; then we have

$$C^2R^2 = 2Mx + N;$$

assuming, for brevity, $M = BE - DC$, $N = E^2 - CF$.

If M be positive, $2Mx + N$ is positive for all values of x greater than $-\frac{N}{2M}$, negative for all others, and continually increases from 0 to ∞, when x increases from $-\frac{N}{2M}$ to ∞. Hence R is impossible for all values of x less than $-\frac{N}{2M}$, possible for all greater values, and increases continually from 0 to ∞, when x increases from $-\frac{N}{2M}$ to ∞. It follows therefore that (fig. 114) represents the locus, where $OM_1 = -\frac{N}{2M}$.

Except when $BE - DC$, or M, is zero, in which case R is invariable, being real when N, or $E^2 - CF$, is greater than zero, and impossible when less. Therefore the locus is two right lines parallel to AB, or has no existence, according as E^2 is > or < CF.

2nd. Let $B^2 - AC$ be negative; then we have

$$C^2R^2 = (B^2 - AC)\{x^2 + 2Mx + N\},$$

where $$M = \frac{BE - DC}{B^2 - AC}, \qquad N = \frac{E^2 - CF}{B^2 - AC};$$

therefore R^2 has the same sign as, and is proportional to,

$$-(x^2 + 2Mx + N) \quad \text{or} \quad M^2 - N - (x - M)^2.$$

Hence, if M^2 be less than N, R^2 is essentially negative, and therefore there is no locus: but if M^2 be greater than N, R^2 is positive for all values of x which make $(x - M)^2$ less than $M^2 - N$, *i. e.* for all values of x between $M - \sqrt{(M^2-N)}$, and $M+\sqrt{(M^2-N)}$, and negative for all other values: also, R^2 is zero when $x = M \pm \sqrt{(M^2 - N)}$, and is greatest when $x = M$. Hence it

appears that fig. 115 represents the locus, where OM_1 and OM_3 are $M \pm \sqrt{(M - N)}$, and $OM_2 = M$.

If $M^2 = N$, the only real value of R is obtained by putting $x' = M$, and therefore the locus is a single point.

3rd. Let $B^2 - AC$ be positive, then R^2 has the same sign as, and is proportional to

$$x^2 + 2Mx + N \quad \text{or} \quad (x - M)^2 + N - M^2.$$

Hence, if M^2 be less than N, R^2 is essentially positive for all values of x, and is least when $x = M$. Therefore (fig. 116) represents the locus. But if M^2 be greater than N, R^2 is negative for all values of x between $M - \sqrt{(M^2 - N)}$ and $M + \sqrt{(M^2 - N)}$, and positive for all other values; also, R^2 is zero when $x = M \pm \sqrt{(M^2 - N)}$, and continually increases, when $(x - M)^2$ increases from $M^2 - N$ to ∞. Therefore (fig. 117) represents the locus.

If $M^2 = N$, $C^2R^2 = (x - M)^2$, and therefore

$$y = -\frac{Bx + E}{C} \pm \frac{x - M}{C},$$

which represents two right lines.

Thus we have traced the form of the locus in all the cases that may occur, and the results thus obtained agree exactly with those previously obtained by different methods.

Of the modifications to be made in the preceding propositions when the co-ordinates are oblique.

412. In Prop. CXXI., instead of putting $x = h + r \cos \theta$, $y = k + r \sin \theta$, we must put $x = h + rm$, $y = k + rn$, where

$$m = \frac{\sin(\omega - \theta)}{\sin \omega}, \qquad n = \frac{\sin \theta}{\sin \omega},$$

ω being the angle of ordination. It is clear that the equation of the line r is

$$\frac{x - h}{m} = \frac{y - k}{n}.$$

Hence the results of this proposition must be stated in the following manner when the co-ordinates are oblique.

(1) The condition necessary in order that (hk) may be the middle point of the chord whose equation is

$$\frac{x-h}{m}=\frac{y-k}{n},$$

is $$(Ah+Bk+D)m+(Bk+Ch+E)n=0.$$

(2) The equation of the diameter of the chords parallel to

$$\frac{x-h}{m}=\frac{y-k}{n};$$

is $$(Am+Bn)x+(Cn+Bm)y+Dm+En=0.$$

(3) If $\frac{x-h}{m}=\frac{y-k}{n}$ be the equation of the tangent at the point (hk), then $$\frac{n}{m}=-\frac{Ah+Bk+D}{Ck+Bh+E}.$$

The results (4), (5), (6), and (7) require no modification.

413. All the propositions which follow Prop. CXXI, except Prop. CXXIII. are equally true for oblique co-ordinates as for rectangular, provided we put m and n for $\cos\theta$ and $\sin\theta$, and $\frac{n}{m}$ for $\tan\theta$.

414. In the case of Prop. CXXIII. the condition necessary, in order that the diameter be parallel to its chords, must be obtained from Art. 91. Let the equation of one of the chords, and the equation of the diameter, be

$$\frac{x-h}{m}=\frac{y-k}{n} \quad \text{or} \quad nx-my=C,$$

$$\frac{x-h'}{m'}=\frac{y-k'}{n'} \quad \text{or} \quad n'x-m'y=C'.$$

Then, by Art. 91, the condition necessary, in order that these two lines be perpendicular, is

$$n'(n+m\cos\omega)+m'(m+n\cos\omega)=0\,\ldots\ldots(1);$$

but, by (2), Art. 412, we have

$$\frac{m'}{n'}=-\frac{Cn+Bm}{Am+Bn}.$$

Hence (1) becomes

$$(Am+Bn)(n+m\cos\omega)=(Cn+Bm)(m+n\cos\omega),$$

which is the condition required, and agrees with the last result of Art. 382. It may be put in the form

$$(A\cos\omega - B)\frac{m^2}{n^2} + (A - C)\frac{m}{n} + B - C\cos\omega = 0,$$

which gives two values of m, and therefore two sets of chords whose diameters are at right angles to them.

Various Problems.

415. ABC is a triangle, whose sides AB, AC, BC, always pass through three given points (hk), $(h'k')$, $(h''k'')$, respectively; and the angular points A and B move along two given right lines OAX, OBY, respectively: to find the locus of C.

Taking OX and OY as co-ordinate axes, let the equations AB, AC, BC be

$$\alpha x + \beta y = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$\alpha x + \beta' y = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$\alpha' x + \beta y = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Observing that, since (1) and (2) meet in the axis of x, and (1) and (3) in the axis of y, (1) and (2) must give the same value of x when $y = 0$, and (1) and (3) the same value of y when $x = 0$.

Since (1), (2), (3) always pass through (hk), $(h'k')$, $(h''k'')$, respectively, we have

$$\alpha h + \beta k = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$\alpha h' + \beta' k' = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$
$$\alpha' h'' + \beta k'' = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

And, if we eliminate α, β, α', β', between (2), (3), (4), (5), (6), the result will represent the locus of the intersection of (2) and (3), *i.e.* the locus of C.

Now, $k'(2) - y(5)$, and $h''(3) - x(6)$ give

$$\alpha(k'x - h'y) = k' - y,$$
$$\beta(h''y - k''x) = h'' - x.$$

Hence, by (4), we have

$$\frac{h(k' - y)}{k'x - h'y} + \frac{k(h'' - x)}{h''y - k''x} = 1,$$

which represents the locus required, which is therefore a line of the second order.

416. To find the locus of the intersection of two tangents drawn at the extremities of a chord of a conic section, supposing that the chord always touches another conic section.

Take the axes of the latter conic section as co-ordinate axes, let its equation be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and let the equation of the former conic section be

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \ldots. (2).$$

Let (hk) be the point of intersection of any two tangents of (2), then the equation of the chord joining the points of contact is

$$(Ah + Bk + D)\, x + (Ck + Bh + E)\, y + Dh + Ek + F = 0 \ldots (3).$$

Now if (3) be a tangent of (1), we have, by Art. 277,

$$(Ah + Bk + D)^2 a^2 + (Ck + Bh + E)^2 b^2 = (Dh + Ek + F)^2,$$

which, being a relation between the co-ordinates, h and k, of the point of intersection of two tangents, drawn at the extremities of any chord of (2) which touches (1), is the equation of the locus required. The locus is, therefore, a line of the second order.

417. The equations of four right lines being

$$ax + by + c = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$a'x + b'y + c' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$\alpha x + \beta y + \gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$\alpha' x + \beta' y + \gamma' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

to shew that the general equation of a line of the second order, which passes through the points of intersection of (1) and (2), of (2) and (3), of (3) and (4), and of (4) and (1), is

$$\lambda\,(ax + by + c)\,(\alpha x + \beta y + \gamma) + \lambda'\,(a'x + b'y + c')\,(\alpha' x + \beta' y + \gamma') = 0 \ldots (5),$$

λ and λ' being arbitrary constants.

Let u, u', v, v' represent the first members of (1), (2), (3), (4), respectively; then the equation

$$\lambda uv + \lambda' u'v' + \mu uv' + \mu' u'v + \nu uu' + \nu' vv' = 0 \ldots. (6),$$

where $\lambda\lambda'$ $\mu\mu'$ $\nu\nu'$ are arbitrary constants, is evidently an equation of the second degree in x and y, and may be made to agree with the general equation of the second degree by giving proper

values to the six arbitrary constants. We may therefore regard (6) as the general equation of the second degree.

Now if (6) be the equation of a curve which passes through the points of intersection of (1) and (2), of (2) and (3), of (3) and (4), and of (4) and (5), it is clear that, when we give x and y such values as make u and u' zero, u' and v zero, v and v' zero, or v' and u zero, (6) must be satisfied; which requires that μ, μ', ν, and ν' shall each be zero. Hence (6) becomes $\lambda uv + \lambda' u'v' = 0$; which coincides with (5). Q. E. D.

418. A conic section and a right line are given, and any quadrilateral is inscribed in the conic section; to shew that, if three sides of the quadrilateral cut the given line in three given points, the fourth side will also cut it in a given point.

Take the given line as axis of x, and any other line as axis of y, and let the equation of the conic section be

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \ldots\ldots (1).$$

Let the equations of the four sides of the quadrilateral be

$$ax + by + 1 = 0 \ldots\ldots\ldots\ldots\ldots (2),$$
$$a'x + b'y + 1 = 0 \ldots\ldots\ldots\ldots\ldots (3),$$
$$\alpha x + \beta y + 1 = 0 \ldots\ldots\ldots\ldots\ldots (4),$$
$$\alpha'x + \beta'y + 1 = 0 \ldots\ldots\ldots\ldots\ldots (5);$$

then, by the preceding problem, the equation of the conic section must be

$$\lambda(ax + by + 1)(\alpha x + \beta y + 1) + \lambda'(a'x + b'y + 1)(\alpha' x + \beta' y + 1) = 0 \ldots (6);$$

and this equation must agree with (1). We have, therefore,

$$A = \lambda a\alpha + \lambda' a'\alpha',$$
$$2D = \lambda(a + \alpha) + \lambda'(a' + \alpha'),$$
$$F = \lambda + \lambda';$$

and from these equations, eliminating λ and λ', we may find α' in terms of a, a', and α. Hence, if a, a', and α be given, α' is also given; but, when (2), (3) and (4) cut the axis of x in three given points, a, a', and α are given; therefore α' is given, and therefore (5) cuts the axis of x in a given point. Q. E. D.

419. In the preceding problem, if three sides of the quadrilateral be parallel to three given lines, the fourth side will also be parallel to a given line.